does not involve the function $\theta(\cdot)$. Thus, one can match the market price $Q_t(T)$ by setting

$$\exp\left(-\int_t^T g(u)\,\mathrm{d}u\right) = \frac{Q_t(T)}{\mathsf{E}_t^R\left[\exp\left(-\int_t^T \hat{r}_u\,\mathrm{d}u\right)\right]}\,.$$

Equivalently, taking logs,

$$-\int_t^T g(u)\,\mathrm{d}u = \log Q_t(T) - \log \mathsf{E}_t^R\left[\exp\left(-\int_t^T \hat{r}_u\,\mathrm{d}u\right)\right]\,.$$

We want this to hold for each $T > t$. Differentiating in T gives

$$g(T) = -\frac{\mathrm{d}}{\mathrm{d}T}\log Q_t(T) + \frac{\mathrm{d}}{\mathrm{d}T}\log \mathsf{E}_t^R\left[\exp\left(-\int_t^T \hat{r}_u\,\mathrm{d}u\right)\right]\,.$$

Taking another derivative in T and using the previous formula for g' yields

$$\kappa(T)[\theta(T) - g(T)] = -\frac{\mathrm{d}^2}{\mathrm{d}T^2}\log Q_t(T) + \frac{\mathrm{d}^2}{\mathrm{d}T^2}\log \mathsf{E}_t^R\left[\exp\left(-\int_t^T \hat{r}_u\,\mathrm{d}u\right)\right]\,.$$

Thus, to fit the yield curve at t, one should choose $\theta(T)$ for $T > t$ by

$$\theta(T) = g(T) - \frac{1}{\kappa(T)}\frac{\mathrm{d}^2}{\mathrm{d}T^2}\log Q_t(T) + \frac{1}{\kappa(T)}\frac{\mathrm{d}^2}{\mathrm{d}T^2}\log \mathsf{E}_t^R\left[\exp\left(-\int_t^T \hat{r}_u\,\mathrm{d}u\right)\right]\,.$$

18.11. Assume the short rate is $r_t = \hat{r}_t + g(t)$, where

$$\mathrm{d}\hat{r} = -\kappa\hat{r}\,\mathrm{d}t + \sigma\,\mathrm{d}B^*\,,$$

for constants κ and σ and $g(\cdot)$ is chosen to fit the current yield curve.

(a) Calculate the forward rates $f_s(u)$ using the Vasicek bond pricing formula.

Solution: The price $P_s(u)$ at s of a discount bond maturing at u is

$$\begin{aligned}\mathsf{E}_s^R\left[\exp\left(-\int_s^u r_t\,\mathrm{d}t\right)\right] &= \exp\left(-\int_s^u g(t)\,\mathrm{d}t\right)\mathsf{E}_s^R\left[\exp\left(-\int_s^u \hat{r}_t\,\mathrm{d}t\right)\right] \\ &= \exp\left(-\int_s^u g(t)\,\mathrm{d}t\right)\mathrm{e}^{Y_s}\,,\end{aligned}$$

where

$$Y_s = -\alpha(\tau) - \beta(\tau)\hat{r}_s$$

with $\tau = u - s$, $\alpha(\tau) = \tau a(\tau)$, $\beta(\tau) = \tau b(\tau)$ and $a(\cdot)$ and $b(\cdot)$ being defined in (18.18), taking $\theta = 0$. Thus,

$$\begin{aligned}
\alpha(\tau) &= -\frac{\sigma^2}{2\kappa^2}\tau + \frac{\sigma^2}{\kappa^3}\left(1 - \mathrm{e}^{-\kappa\tau}\right) - \frac{\sigma^2}{4\kappa^3}\left(1 - \mathrm{e}^{-2\kappa\tau}\right) , \\
\beta(\tau) &= \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa\tau}\right) .
\end{aligned}$$

It follows that the forward rate $f_s(u)$ is

$$\begin{aligned}
-\frac{\mathrm{d}\log P_s(u)}{\mathrm{d}u} &= g(u) + \alpha'(\tau) + \beta'(\tau)\hat{r}_s \\
&= g(u) - \frac{\sigma^2}{2\kappa^2} + \frac{\sigma^2}{\kappa^2}\mathrm{e}^{-\kappa\tau} - \frac{\sigma^2}{2\kappa^2}\mathrm{e}^{-2\kappa\tau} + \mathrm{e}^{-\kappa\tau}\hat{r}_s .
\end{aligned}$$

(b) Calculate $\alpha_s(u)$ and $\sigma_s(u)$ such that, as s changes,

$$\mathrm{d}f_s(u) = \alpha_s(u)\,\mathrm{d}s + \sigma_s(u)\,\mathrm{d}B^*_s .$$

Solution: Fixing u, the forward rate evolves for $s < u$ as

$$\begin{aligned}
\mathrm{d}f_s(u) &= \frac{\sigma^2}{\kappa}\mathrm{e}^{-\kappa\tau}\,\mathrm{d}s - \frac{\sigma^2}{\kappa}\mathrm{e}^{-2\kappa\tau}\,\mathrm{d}s + \kappa\mathrm{e}^{-\kappa\tau}\hat{r}_s\,\mathrm{d}s + \mathrm{e}^{-\kappa\tau}\,\mathrm{d}\hat{r}_s \\
&= \left[\frac{\sigma^2}{\kappa}\mathrm{e}^{-\kappa\tau} - \frac{\sigma^2}{\kappa}\mathrm{e}^{-2\kappa\tau} + \kappa\mathrm{e}^{-\kappa\tau}\hat{r}_s\right]\mathrm{d}s - \kappa\mathrm{e}^{-\kappa\tau}\hat{r}_s\,\mathrm{d}s + \sigma\mathrm{e}^{-\kappa\tau}\,\mathrm{d}B^*_s \\
&= \frac{\sigma^2}{\kappa}\left(\mathrm{e}^{-\kappa\tau} - \mathrm{e}^{-2\kappa\tau}\right)\mathrm{d}s + \sigma\mathrm{e}^{-\kappa\tau}\,\mathrm{d}B^*_s .
\end{aligned}$$

Thus, the drift is

$$\alpha_s(u) = \frac{\sigma^2}{\kappa}\left(\mathrm{e}^{-\kappa(u-s)} - \mathrm{e}^{-2\kappa(u-s)}\right) ,$$

and the volatility is

$$\sigma_s(u) = \sigma\mathrm{e}^{-\kappa(u-s)} .$$

(c) Prove that

$$\alpha_s(u) = \sigma_s(u) \int_s^u \sigma_s(t)\,\mathrm{d}t\,.$$

Solution: We have

$$\begin{aligned}
\sigma_s(u) \int_s^u \sigma_s(t)\,\mathrm{d}t &= \sigma \mathrm{e}^{-\kappa(u-s)} \int_s^u \sigma \mathrm{e}^{-\kappa(t-s)}\,\mathrm{d}t \\
&= \frac{\sigma^2}{\kappa} \mathrm{e}^{-\kappa(u-s)} \left(1 - \mathrm{e}^{-\kappa(u-s)}\right) \\
&= \alpha_s(u)\,.
\end{aligned}$$

18.12. Assume the short rate is $r_t = \hat{r}_t + g(t)$, where

$$\mathrm{d}\hat{r} = -\kappa \hat{r}\,\mathrm{d}t + \sigma\,\mathrm{d}B^*\,,$$

and $g(\cdot)$ is chosen to fit the current yield curve.

(a) Consider a forward contract maturing at T on a discount bond maturing at $u > T$. Let F_t denote the forward price for $t \leq T$. What is the volatility of $\mathrm{d}F_t/F_t$?

Solution: The forward price is $F_t = P_t(u)/P_t(T)$. The volatility of $\mathrm{d}F/F$ is the volatility of $\mathrm{d}\log F$, so it is the volatility of $\mathrm{d}\log P_t(u) - \mathrm{d}\log P_t(T)$. From the previous exercise, the price $P_t(x)$ at t of a discount bond maturing at x is

$$\exp\left(-\int_t^x g(s)\,\mathrm{d}s\right) \mathrm{e}^{Y_t(x)}\,,$$

where

$$Y_t(x) = -\alpha(x-t) - \beta(x-t)\hat{r}_t$$

with

$$\begin{aligned}
\alpha(\tau) &= -\frac{\sigma^2}{2\kappa^2}\tau + \frac{\sigma^2}{\kappa^3}\left(1 - \mathrm{e}^{-\kappa\tau}\right) - \frac{\sigma^2}{4\kappa^3}\left(1 - \mathrm{e}^{-2\kappa\tau}\right)\,, \\
\beta(\tau) &= \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa\tau}\right)\,.
\end{aligned}$$

The volatility of $\mathrm{d}\log P_t(u) - \mathrm{d}\log P_t(T)$ is the volatility of $\mathrm{d}Y_t(u) - \mathrm{d}Y_t(T)$, which is the volatility of

$$-[\beta(u-t) - \beta(T-t)]\mathrm{d}\hat{r}_t\,.$$

Hence, the volatility is

$$\frac{\sigma}{\kappa}\left(\mathrm{e}^{-\kappa(T-t)} - \mathrm{e}^{-\kappa(u-t)}\right) = \frac{\sigma}{\kappa}\left(\mathrm{e}^{-\kappa T} - \mathrm{e}^{-\kappa u}\right)\mathrm{e}^{\kappa t}\,.$$

(b) What is the average volatility between 0 and T of $\mathrm{d}F_t/F_t$ in the sense of (17.9)?

Solution: The average volatility in the sense of (17.9) is

$$\begin{aligned}
\sigma_{\text{avg}} &= \frac{\sigma}{\kappa}\left(\mathrm{e}^{-\kappa T} - \mathrm{e}^{-\kappa u}\right)\sqrt{\frac{1}{T}\int_0^T \mathrm{e}^{2\kappa t}\,\mathrm{d}t} \\
&= \frac{\sigma}{\kappa}\left(\mathrm{e}^{-\kappa T} - \mathrm{e}^{-\kappa u}\right)\sqrt{\frac{\mathrm{e}^{2\kappa T} - 1}{2\kappa T}}
\end{aligned}$$

(c) Consider a call option maturing at T on a discount bond that matures at $u > T$. Derive a formula for the value of the call option at date 0.

Solution: Because the risk-free rate is stochastic, we need to use Merton's (equivalently, Black's) formula. The value at date 0 is

$$P_0(u)\,\mathrm{N}(d_1) - \mathrm{e}^{-yT}K\,\mathrm{N}(d_2)\,,$$

where

$$\begin{aligned}
d_1 &= \frac{\log(P_0(u)/K) + \left(y + \frac{1}{2}\sigma_{\text{avg}}^2\right)T}{\sigma_{\text{avg}}\sqrt{T}}\,, \\
d_2 &= d_1 - \sigma_{\text{avg}}\sqrt{T}\,,
\end{aligned}$$

and $y = -\log P_0(T)/T$.

Chapter 19

Perpetual Options and the Leland Model

19.1. Calculate the value of a perpetual call that is knocked out when the underlying asset price falls to a boundary s_L. What is the optimal exercise boundary for the call? How does the optimal boundary compare to the optimal boundary when there is no knock-out feature and why? Answer the same questions for a perpetual put that is knocked out when the underlying asset price rises to a boundary s_H.

Solution: Let $s_* > K$ denote a possible exercise boundary for the call. If exercised at s_*, the owner of the call receives $s_* - K$ at the hitting time of s_*, unless the asset price hits s_L first, in which case the owner receives zero. For $s_L < S_t < s_*$, the value of the call is $aS_t^{-\gamma} + bS_t^{\beta}$, where $-\gamma$ and β are the two roots of the characteristic equation and where a and b satisfy

$$as_L^{-\gamma} + bs_L^{\beta} = 0\,,$$

$$as_*^{-\gamma} + bs_*^{\beta} = s_* - K\,.$$

The first of these equations gives us $a = -bs_L^{\beta+\gamma}$. Substitute this into the second and solve for b to obtain

$$
\begin{aligned}
a &= \frac{s_* - K}{s_*^{-\gamma} - s_L^{-\beta-\gamma} s_*^{\beta}}\,, \\
b &= \frac{s_* - K}{s_*^{\beta} - s_L^{\beta+\gamma} s_*^{-\gamma}}\,.
\end{aligned}
$$

We can find s_* by smooth pasting. The smooth pasting condition is

$$\frac{\mathrm{d}}{\mathrm{d}s} as^{-\gamma} + bs^{\beta}\bigg|_{s=s_*} = 1\,.$$

Thus,

$$-\gamma a s_*^{-\gamma-1} + \beta b s_*^{\beta-1} = 1\,.$$

Multiply by $s_*/(s_* - K)$ and substitute for a and b to obtain

$$-\frac{\gamma}{1 - s_L^{-\beta-\gamma} s_*^{\beta+\gamma}} + \frac{\beta}{1 - s_L^{\beta+\gamma} s_*^{-\beta-\gamma}} = \frac{s_*}{s_* - K}\,.$$

The solution s_* of this equation is the optimal exercise boundary. We can simplify a bit by setting $z = s_*/s_L$. The equation becomes

$$-\frac{\gamma}{1 - z^{\beta+\gamma}} + \frac{\beta}{1 - z^{-\beta-\gamma}} = \frac{z}{z - K/S_L}\,.$$

As far as I know, this equation for z must be solved numerically. With $r = 0.02$, $\delta = 0.03$, and $\sigma = 0.2$, we obtain $\beta = 2$ and $\gamma = 0.5$. With $K = 50$ and $s_L = 40$, we get an optimal boundary $s_* = 84.26$. In contrast, the optimal boundary with no knock-out feature is $\beta K/(\beta - 1) = 100$. This illustrates the general feature that the knock-out feature lowers the optimal exercise boundary—it is optimal to exercise earlier to avoid the risk of being knocked out.

The analysis of a put is very similar. Given an exercise boundary s_*, the value is $as^{-\gamma} + bs^{\beta}$

for $s_H > s > s^*$, where a and b are defined by

$$as_*^{-\gamma} + bs_*^{\beta} = K - s_*\,,$$

$$as_H^{-\gamma} + bs_H^{\beta} = 0\,.$$

The bottom equation gives $a = -bs_H^{\beta+\gamma}$. Substitute this into the first and solve to obtain

$$a = \frac{K - s_*}{s_*^{-\gamma} - s_H^{-\beta-\gamma}s_*^{\beta}}\,,$$

$$b = \frac{K - s_*}{s_*^{\beta} - s_H^{\beta+\gamma}s_*^{-\gamma}}\,.$$

We again find s_* by using smooth pasting. The smooth pasting condition now is

$$\frac{\mathrm{d}}{\mathrm{d}s}as^{-\gamma} + bs^{\beta}\bigg|_{s=s_*} = -1\,.$$

Thus,

$$-\gamma as_*^{-\gamma-1} + \beta bs_*^{\beta-1} = -1\,.$$

Multiply by $s_*/(K - s_*)$ and substitute for a and b to obtain

$$-\frac{\gamma}{1 - s_H^{-\beta-\gamma}s_*^{\beta+\gamma}} + \frac{\beta}{1 - s_H^{\beta+\gamma}s_*^{-\beta-\gamma}} = \frac{s_*}{s_* - K}\,.$$

This is the same equation as for a call, but with s_H replacing s_L. Putting $z = s_*/s_H$, the equation is

$$-\frac{\gamma}{1 - z^{\beta+\gamma}} + \frac{\beta}{1 - z^{-\beta-\gamma}} = \frac{z}{z - K/S_H}\,.$$

For a call, we looked for a solution $z > 1$. Now, we are interested in a solution $z < 1$.

With $s_H = 60$, and the other parameters the same as before, the optimal boundary is 21.48. In contrast, without the knock-out feature, the optimal boundary is $\gamma K/(1+\gamma) = 16.67$. Again, the knock-out feature causes it to be optimal to exercise the option earlier, to avoid the risk of being knocked out.

The optimal boundaries can also be found by directly optimizing over s_* instead of using smooth pasting. For the optimal boundaries to be independent of the current value of S_t, we must have $a'(s_*) = b'(s_*) = 0$. This first-order condition holds at the s_* found by smooth pasting. This can be verified numerically.

19.2. Explain how (19.6), (19.17), and (19.19) are used to compute the values (19.22)–(19.25) of corporate claims in the Leland model.

Solution:

Debt: The first term in (19.22) is the cash flow $(1-\tau_i)c$ received until default times the value of receiving a cash flow of \$1 until default—given in (19.17). The second term is the cash flow $(1-\alpha)(1-\tau_{\text{eff}})x_D$ received at default times the value of receiving \$1 at default—given in (19.6).

Equity: The first term in (19.23) is the value of receiving the cash flow $(1-\tau_{\text{eff}})\delta X_t$ until default—given in (19.19). The second term is the cash flow $(1-\tau_{\text{eff}})c$ that is paid until default times the value of receiving a cash flow of \$1 until default—given in (19.17).

Bankruptcy Costs: (19.24) is the bankrupcty cost $\alpha(1-\tau_{\text{eff}})x_D$ times the value of receiving \$1 at default—given in (19.6).

Taxes: The first term in (19.25) is the value of receiving the cash flow $\tau_{\text{eff}}X_t$ forever. The second term is the cash flow $(\tau_i-\tau_{\text{eff}})c$ times the value of receiving a cash flow of \$1 until default—given in (19.17).

19.3. Derive the value of tax payments (19.41) and the value of issuance costs (19.42) in the model of dynamic capital structure.

Solution: The tax payments (using information on page 496) are $\tau_{\text{eff}}\delta X_t$ forever, $(\tau_i - \tau_{\text{eff}})c$ until either default or refinancing occurs, $(\tau_i - \tau_{\text{eff}})c\phi_R$ after refinancing until default or a second refinancing occurs, and so forth. The value of receiving $\tau_{\text{eff}}\delta X_t$ forever is $\tau_{\text{eff}}X_0$, which is just τ_{eff} since we are normalizing $X_0 = 1$.

Let $\hat{H}_0$ denote the value of receiving $(\tau_i - \tau_{\text{eff}})c$ until either default or refinancing occurs, $(\tau_i - \tau_{\text{eff}})c\phi_R$ after refinancing until default or a second refinancing occurs, and so forth. Then

$$\hat{H}_0 = (\tau_i - \tau_{\text{eff}})\frac{c}{r}(1 - P_0^D - P_0^R)\left[1 + \phi_R P_0^R + (\phi_R P_0^R)^2 + \cdots\right] = \frac{(\tau_i - \tau_{\text{eff}})(c/r)(1 - P_0^D - P_0^R)}{1 - \phi_R P_0^R}\,.$$

Combining the two parts of the tax payments gives a value of $H_0 = \hat{H}_0 + \tau_{\text{eff}}$, which verifies (19.41).

Issuance costs are qD_0 at date 0, $q\phi_R D_0$ at the time of refinancing if refinancing occurs before default, and so forth. Excluding date–0 issuance costs, the value of future issuance costs is

$$qD_0\left[\phi_R P_0^R + (\phi_R P_0^R)^2 + \cdots\right] = \frac{qD_0\phi_R P_0^R}{1 - \phi_R P_0^R}\,.$$

19.4. Consider the value $f(S_t)$ of receiving a cash flow c per unit of time until the asset price S hits a boundary s^*. Assume the dividend yield, volatility, and risk-free rate are constant. The value is derived in Section 19.2 by reducing the valuation problem to valuing the receipt of \$1 at the hitting time. The valuation can be done more directly as follows:

(a) Write down the ODE that the function f satisfies, taking into account that part of the return from the asset is the cash flow $c\,\mathrm{d}t$.

Solution: The rate of return is

$$\begin{aligned}\frac{\mathrm{d}f + c\,\mathrm{d}t}{f} &= \frac{1}{f}\left[f'(S)\,\mathrm{d}S + \frac{1}{2}f''(S)(\mathrm{d}S)^2 + c\,\mathrm{d}t\right]\\ &= \frac{1}{f}\left[f'(S)(r-\delta)S\,\mathrm{d}t + f'(S)\sigma S\,\mathrm{d}B^* + \frac{1}{2}f''(S)S^2\sigma^2\,\mathrm{d}t + c\,\mathrm{d}t\right]\end{aligned}$$

Equating the expected rate of return under the risk-neutral probability to the risk-free rate gives

$$(r-\delta)Sf'(S)+\frac{1}{2}\sigma^2S^2f''(S)+c=rf\,.$$

(b) Show that c/r satisfies the ODE.

Solution: Setting $f(s)=c/r$ gives $f'=f''=0$. Substituting into the ODE, the left-hand side is c, and the right-hand side is $rf=c$.

(c) Show that

$$\frac{c}{r}+as^{-\gamma}+bs^{\beta}$$

satisfies the ODE for any constants a and b, where $-\gamma$ and β are the roots of the quadratic equation (19.3). **In the text, the displayed expression above is erroneously written as $c/r+as^{-\gamma}+bs^{\gamma}$.** (The ODE is called nonhomogeneous, because it includes the constant term c—a term that is not linear in f. The general method for solving a nonhomogeneous version of (19.2) is to value the nonhomogeneous part as a cash flow to be received forever and then to add that value to the general solution (19.4) of the homogeneous part.)

Solution: Now we have

$$f'(s)=-\gamma as^{-\gamma-1}+\beta bs^{\beta-1}\,,$$
$$f''(s)=\gamma(\gamma+1)as^{-\gamma-2}+\beta(\beta-1)bs^{\beta-2}\,.$$

Substituting into the ODE, the left-hand side is

$$\begin{aligned}(r-\delta)s&\Big[-\gamma as^{-\gamma-1}+\beta bs^{\beta-1}\Big]+\frac{1}{2}\sigma^2s^2\Big[\gamma(\gamma+1)as^{-\gamma-2}+\beta(\beta-1)bs^{\beta-2}\Big]+c\\&=(r-\delta)\Big[-\gamma as^{-\gamma}+\beta bs^{\beta}\Big]+\frac{1}{2}\sigma^2\Big[\gamma(\gamma+1)as^{-\gamma}+\beta(\beta-1)bs^{\beta}\Big]+c\\&=\Big[-\gamma(r-\delta)+\frac{1}{2}\sigma^2\gamma(\gamma+1)\Big]as^{-\gamma}+\Big[\beta(r-\delta)+\frac{1}{2}\sigma^2\beta(\beta-1)\Big]bs^{\beta}+c\,.\end{aligned}$$

The right-hand side is

$$rf = c + ras^{-\gamma} + rbs^{\beta}\,.$$

The terms involving $as^{-\gamma}$ on both sides match if $-\gamma$ is a root of the characteristic equation. Likewise the terms involving bs^{β} on both sides match if β is a root of the characteristic equation. Cancelling those terms leaves $c = c$.

(d) Use boundary conditions to calculate a and b in the two cases (i) the cash flow terminates when S_t first falls to s^*, and (ii) the cash flow terminates when S_t first rises to s^*.

Solution: In case (i), we have $\lim_{s\to\infty} f(s) = c/r$ (the value of receiving the coupon forever), which implies $\beta = 0$. Also, $f(s^*) = 0$, so $c/r + a(s^*)^{-\gamma} = 0$, which implies $a = -(c/r)(s^*)^{\gamma}$. Consequently, the value of the security is

$$\frac{c}{r}\left[1 - \left(\frac{s^*}{s}\right)^{\gamma}\right].$$

In case (ii), we have $\lim_{s\to 0} f(s) = c/r$, which implies $\gamma = 0$. Also, $f(s^*) = 0$, so $c/r + b(s^*)^{\beta} = 0$, which implies $b = -(c/r)(s^*)^{-\beta}$. Consequently, the value of the security is

$$\frac{c}{r}\left[1 - \left(\frac{s}{s^*}\right)^{\beta}\right].$$

19.5. Repeat the previous exercise for the value of receiving κS_t until S hits a boundary, for a constant $\kappa > 0$. In lieu of part (b) of the previous exercise, show that $f(s) = \kappa s/\delta$ satisfies the ODE.

Solution: To derive the ODE, note that the rate of return is

$$\begin{aligned}
\frac{\mathrm{d}f + \kappa S\,\mathrm{d}t}{f} &= \frac{1}{f}\left[f'(S)\,\mathrm{d}S + \frac{1}{2}f''(S)(\mathrm{d}S)^2 + \kappa S\,\mathrm{d}t\right] \\
&= \frac{1}{f}\left[f'(S)(r-\delta)S\,\mathrm{d}t + f'(S)\sigma S\,\mathrm{d}B^* + \frac{1}{2}f''(S)S^2\sigma^2\,\mathrm{d}t + \kappa S\,\mathrm{d}t\right]
\end{aligned}$$