Solution: From Itô's formula,

$$\begin{aligned}\frac{\mathrm{d}W}{W} &= \frac{\mathrm{d}g}{g} + \frac{\mathrm{d}M^{-1/\rho}}{M^{-1/\rho}} \\ &= \frac{g'}{g}\,\mathrm{d}t + \frac{-(1/\rho)M^{-1-1/\rho}\,\mathrm{d}M + (1/2)(1/\rho)(1+1/\rho)M^{-2-1/\rho}(\mathrm{d}M)^2}{M^{-1/\rho}}\,.\end{aligned}$$

The stochastic part of this comes from

$$\frac{-(1/\rho)M^{-1-1/\rho}\,\mathrm{d}M}{M^{-1/\rho}} = -\frac{1}{\rho}\frac{\mathrm{d}M}{M} = \frac{r}{\rho}\,\mathrm{d}t + \frac{1}{\rho}\lambda_p'\,\mathrm{d}B\,.$$

Hence the stochastic part of the portfolio return must be

$$\frac{1}{\rho}\lambda_p'\,\mathrm{d}B\,,$$

implying

$$\pi'\sigma = \frac{1}{\rho}\lambda_p' = \frac{1}{\rho}(\mu - r\iota)'\Sigma^{-1}\sigma\,.$$

Postmultiplying by $\sigma'\Sigma^{-1}$ shows that this has the unique solution

$$\pi' = \frac{1}{\rho}(\mu - r\iota)'\Sigma^{-1}\,.$$

14.7. This exercise demonstrates the equivalence between the intertemporal and static budget constraints in the presence of labor income when the investor can borrow against the income, as asserted in Section 14.3. Let M be an SDF process and Y a labor income process. Assume

$$\mathsf{E}\left[\int_0^T M_t|Y_t|\,\mathrm{d}t\right] < \infty$$

for each finite T. The intertemporal budget constraint is

$$\mathrm{d}W = rW\,\mathrm{d}t + \phi'(\mu - r\iota)\,\mathrm{d}t + Y\,\mathrm{d}t - C\,\mathrm{d}t + \phi'\sigma\,\mathrm{d}B\,. \tag{14.34}$$

(a) Suppose that (C, W, ϕ) satisfies the intertemporal budget constraint (14.34), $C \geq 0$, and the nonnegativity constraint (14.8) holds.

(i) Suppose the horizon is finite. Show that (C, W) satisfies the static budget constraint

$$W_0 + \mathsf{E}\left[\int_0^T M_t Y_t \,\mathrm{d}t\right] \geq \mathsf{E}\left[\int_0^T M_t C_t \,\mathrm{d}t + M_T W_T\right] \tag{14.35}$$

by showing that

$$\int_0^t M_s(C_s - Y_s)\,\mathrm{d}s + M_t W_t$$

is a supermartingale.

Hint: Show that it is a local martingale and at least as large as the martingale $-X_t$, where

$$X_t = \mathsf{E}_t\left[\int_0^T M_s Y_s \,\mathrm{d}s\right].$$

This implies the supermartingale property (Appendix A.13.)

Solution: The differential of

$$\int_0^t M_s(C_s - Y_s)\,\mathrm{d}s + M_t W_t$$

is

$$\begin{aligned}
&M(C - Y)\,\mathrm{d}t + M\,\mathrm{d}W + W\,\mathrm{d}M + (\mathrm{d}M)(\mathrm{d}W) \\
&\quad = rMW\,\mathrm{d}t + M\phi'(\mu - r\iota)\,\mathrm{d}t + M\phi'\sigma\,\mathrm{d}B - rMW\,\mathrm{d}t - MW\lambda'\,\mathrm{d}B - M\phi'\sigma\lambda\,\mathrm{d}t \\
&\quad = M\phi'\sigma\,\mathrm{d}B - MW\lambda'\,\mathrm{d}B\,.
\end{aligned}$$

Thus, it is a local martingale. By the nonnegative wealth constraint (14.8) and the nonnegativity of C,

$$\int_0^t M_s C_s \,\mathrm{d}s + M_t W_t + \mathsf{E}_t\left[\int_t^T M_s Y_s \,\mathrm{d}s\right] \geq 0\,.$$

This implies

$$\int_0^t M_s(C_s - Y_s)\,\mathrm{d}s + M_t W_t \geq -\mathsf{E}_t\left[\int_0^T M_s Y_s\,\mathrm{d}s\right] .$$

Hence,

$$\int_0^t M_s(C_s - Y_s)\,\mathrm{d}s + M_t W_t$$

is a supermartingale, which implies

$$W_0 \geq \mathsf{E}\left[\int_0^T M_s(C_s - Y_s)\,\mathrm{d}s + M_T W_T\right] .$$

Rearranging produces the static budget constraint.

(ii) Suppose the horizon is infinite and $\lim_{T\to\infty} \mathsf{E}[M_T W_T] \geq 0$. Assume $Y \geq 0$. Show that the static budget constraint

$$W_0 + \mathsf{E}\left[\int_0^\infty M_t Y_t\,\mathrm{d}t\right] \geq \mathsf{E}\left[\int_0^\infty M_t C_t\,\mathrm{d}t\right]$$

holds.

Solution: Taking the limit of the finite-horizon static budget constraint as $T \to \infty$, using the nonnegativity of C, Y and M and the Monotone Convergence Theorem yields

$$W_0 + \mathsf{E}\left[\int_0^\infty M_t Y_t\,\mathrm{d}t\right] \geq \mathsf{E}\left[\int_0^\infty M_t C_t\,\mathrm{d}t\right] + \lim_{T\to\infty} \mathsf{E}[M_T W_T] \geq \mathsf{E}\left[\int_0^\infty M_t C_t\,\mathrm{d}t\right] .$$

(b) Suppose the horizon is finite, markets are complete, $C \geq 0$, and (C, W) satisfies the static budget constraint (14.35) as an equality. Show that there exists ϕ such that (C, W, ϕ) satisfies the intertemporal budget constraint (14.34).

Solution: The proof is the same as in Section 13.5, replacing C by $C - Y$. Specifically, defining

$$W_t = \mathsf{E}_t\left[\int_t^T \frac{M_s}{M_t}(C_s - Y_s)\,\mathrm{d}s + \frac{M_T}{M_t} W_T\right] ,$$

it follows that

$$\int_0^t M_s(C_s - Y_s)\,\mathrm{d}s + M_t W_t$$

is a martingale. Use the martingale representation theorem and choose ϕ to match $\phi'\sigma\,\mathrm{d}B$ to the stochastic part of $\mathrm{d}W$ as in Section 13.5.

Chapter 15

Continuous-Time Topics

15.1. Assume there is a representative investor with constant relative risk aversion ρ. Assume aggregate consumption C satisfies

$$\frac{\mathrm{d}C}{C} = \alpha(X)\,\mathrm{d}t + \theta(X)'\,\mathrm{d}B$$

for functions α and θ, where X is the Markov process (13.50).

(a) Explain why the market price-dividend ratio is a function of X_t.

Solution: The market price-dividend ratio at date t is

$$\mathsf{E}_t \int_t^\infty \frac{M_u C_u}{M_t C_t}\,\mathrm{d}u = \mathsf{E}_t \int_t^\infty \mathrm{e}^{-\delta(u-t)} \left(\frac{C_u}{C_t}\right)^{1-\rho} \mathrm{d}u\,.$$

Because X_t is a sufficient statistic for the distribution of C_u/C_t for $u \geq t$, this is a function of X_t.

(b) Denote the market price-dividend ratio by $f(X_t)$. Explain why the market risk premium is

$$\rho\theta'\theta + \rho\theta' \left(\sum_{j=1}^{\ell} \left.\frac{\partial \log f(x)}{\partial x_j}\right|_{x=X_t} \nu_j \right).$$

How does this compare to the geometric Brownian motion model of consumption in Exercise 13.2?

Solution: Let P_t denote the price of the market portfolio, so we have $P_t/C_t = f(X_t)$ and therefore $P_t = C_t f(X_t)$. This implies

$$\frac{\mathrm{d}P}{P} = \frac{\mathrm{d}C}{C} + \frac{\mathrm{d}f(X_t)}{f(X_t)} + \left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}f(X_t)}{f(X_t)}\right).$$

Also, from Exercise 13.2(c),

$$\frac{\mathrm{d}M}{M} = -\delta\,\mathrm{d}t - \rho\frac{\mathrm{d}C}{C} + \frac{\rho(1+\rho)}{2}\left(\frac{\mathrm{d}C}{C}\right)^2.$$

It follows that the market risk premium is

$$\begin{aligned}
-\left(\frac{\mathrm{d}P}{P}\right)\left(\frac{\mathrm{d}M}{M}\right) &= \rho\left(\frac{\mathrm{d}C}{C}\right)^2 + \rho\left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}f(X_t)}{f(X_t)}\right) \\
&= \rho\theta'\theta\,\mathrm{d}t + \rho(\theta'\,\mathrm{d}B)\left(\sum_{j=1}^{\ell}\frac{f_{x_j}(X_t)}{f(X_t)}\nu_j'\,\mathrm{d}B\right) \\
&= \rho\theta'\theta\,\mathrm{d}t + \rho\sum_{j=1}^{\ell}\frac{f_{x_j}(X_t)}{f(X_t)}\theta'\nu_j\,\mathrm{d}t\,.
\end{aligned}$$

In the geometric Brownian motion model of consumption, the market risk premium is just $\rho\theta'\theta$.

15.2. Adopt the assumptions of Part (a) of Exercise 13.2. Assume $\mathrm{e}^{rt}M_t$ is a martingale.

(a) Using Girsanov's theorem, show that

$$\frac{\mathrm{d}D}{D} = (\mu - \sigma\lambda)\,\mathrm{d}t + \sigma\,\mathrm{d}B^*\,,$$

where B^* is a Brownian motion under the risk neutral probability associated with M.

Solution: Girsanov's theorem states that

$$\mathrm{d}B^* = \mathrm{d}B - \left(\frac{\mathrm{d}M}{M}\right)(\mathrm{d}B) = \mathrm{d}B + \lambda\,\mathrm{d}t$$

defines a Brownian motion under the risk-neutral probability. Substituting B^* for B in the equation for $\mathrm{d}D$ gives the result.

(b) Calculating under the risk neutral probability, show that the asset price is

$$P_t \stackrel{\text{def}}{=} \frac{D_t}{r + \sigma\lambda - \mu}\,.$$

Verify that the expected rate of return of the asset under the risk neutral probability is the risk-free rate.

Solution: The asset price is

$$\begin{aligned}
\mathsf{E}_t^* \int_t^\infty \mathrm{e}^{-r(u-t)} D_u \,\mathrm{d}u &= \int_t^\infty \mathrm{e}^{-r(u-t)} \mathsf{E}_t^*[D_u]\,\mathrm{d}u \\
&= \int_t^\infty \mathrm{e}^{-r(u-t)} \mathrm{e}^{(\mu-\sigma\lambda)(u-t)} D_t \,\mathrm{d}u \\
&= \frac{D_t}{r+\sigma\lambda-\mu}\,.
\end{aligned}$$

Define $\delta = r + \sigma\lambda - \mu$, so we have $D/P = \delta$. Then,

$$\frac{\mathrm{d}P + D\,\mathrm{d}t}{P} = \frac{\mathrm{d}P}{P} + \delta\,\mathrm{d}t = \frac{\mathrm{d}D}{D} + \delta\,\mathrm{d}t = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*\,.$$

(c) Define $\delta = r + \sigma\lambda - \mu$, so we have $D/P = \delta$ (in other words, δ is the dividend yield). Verify that

$$\frac{\mathrm{d}P}{P} = (r-\delta)\,\mathrm{d}t + \sigma\,\mathrm{d}B^*\,.$$

Solution: See the previous part.

(d) Write down the fundamental PDE—which is here actually an ODE—for the asset value $P_t = f(D_t)$. Verify that the ODE is satisfied by $f(D) = D/(r + \sigma\lambda - \mu)$.

Solution: Write the rate of return as

$$\begin{aligned}\frac{\mathrm{d}f + D\,\mathrm{d}t}{f} &= \frac{1}{f}\left[f'\,\mathrm{d}D + \frac{1}{2}f''(\mathrm{d}D)^2 + D\,\mathrm{d}t\right] \\ &= \frac{1}{f}\left[f'(\mu - \sigma\lambda)D\,\mathrm{d}t + f'\sigma D\,\mathrm{d}B^* + \frac{1}{2}f''\sigma^2 D^2\,\mathrm{d}t + D\,\mathrm{d}t\right]\end{aligned}$$

Equate the expected rate of return under the risk-neutral probability to $r\,\mathrm{d}t$. This gives the ODE

$$f'(\mu - \sigma\lambda)D + \frac{1}{2}f''\sigma^2 D^2 + D = rf\,.$$

For the given f, we have $f' = 1/(r + \sigma\lambda - \mu)$ and $f'' = 0$. Substituting into the ODE, it simplifies to

$$\frac{(\mu - \sigma\lambda)D}{r + \sigma\lambda - \mu} + D = \frac{rD}{r + \sigma\lambda - \mu}\,,$$

and this equation is true.

15.3. Adopt the assumptions of Section 15.2. Assume the investor constant relative risk aversion ρ. Define optimal consumption C and terminal wealth W_T from the first-order conditions (14.7), and define W_t from (14.5).

(a) Show that $W_t = M_t^{-1/\rho} f(t, X_t)$ for some function f.

Solution: Suppose the investor maximizes

$$\int_0^T \mathrm{e}^{-\delta t}\frac{1}{1-\rho}C_t^{1-\rho}\,\mathrm{d}t + \frac{b}{1-\rho}W_T^{1-\rho}\,.$$

The first-order conditions give us $C_t = \left(\gamma \mathrm{e}^{\delta t} M_t\right)^{-1/\rho}$ and $W_T = (\gamma M_T/b)^{-1/\rho}$. Therefore, the optimal wealth is

$$\begin{aligned}&\mathsf{E}_t\left[\int_t^T \frac{M_u}{M_t}\left(\gamma \mathrm{e}^{\delta u} M_u\right)^{-1/\rho}\,\mathrm{d}u + \frac{M_T}{M_t}(\gamma M_T/b)^{-1/\rho}\right] \\ &\qquad = M_t^{-1/\rho}\mathsf{E}_t\left[\int_t^T \left(\gamma \mathrm{e}^{\delta u}\right)^{-1/\rho}\left(\frac{M_u}{M_t}\right)^{(\rho-1)/\rho}\,\mathrm{d}u + (\gamma/b)^{-1/\rho}\left(\frac{M_T}{M_t}\right)^{(\rho-1)/\rho}\right]\,.\end{aligned}$$

The conditional expectation here is a function of t and X_t, because X_t is a sufficient statistic for the distribution of M_u/M_t for $u > t$.

(b) Derive a PDE for f. **Students may try to derive the PDE from (15.3). There is a mistake in (15.3). The mistake is noted in the errata and also discussed below.**

Solution: We use the fact that

$$\int_0^t M_u C_u \,\mathrm{d}u + M_t W_t = \int_0^t \left(\gamma e^{\delta u}\right)^{-1/\rho} M_u^{1-1/\rho} \,\mathrm{d}u + M_t^{1-1/\rho} f(t, X_t)$$

is a martingale. Its differential is

$$\left(\gamma e^{\delta t}\right)^{-1/\rho} M_t^{1-1/\rho} \,\mathrm{d}t + M_t^{1-1/\rho} \,\mathrm{d}f(t, X_t) + f(t, X_t)\,\mathrm{d}M_t^{1-1/\rho} + \left(\mathrm{d}M_t^{1-1/\rho}\right)(\mathrm{d}f(t, X_t))\,. \quad (*)$$

Use Exercise 12.2(c) and the dynamics of M to obtain

$$\frac{\mathrm{d}M^{1-1/\rho}}{M^{1-1/\rho}} = \frac{\rho - 1}{\rho}\Big(- r\,\mathrm{d}t - \lambda'\,\mathrm{d}B\Big) + \frac{1-\rho}{2\rho^2}\lambda'\lambda\,\mathrm{d}t\,.$$

We want to equate the $\mathrm{d}t$ terms of $(*)$ to zero. We can cancel the $M_t^{1-1/\rho}$ throughout. This yields

$$\left(\gamma e^{\delta t}\right)^{-1/\rho} + f_t + \sum_{i=1}^{\ell} f_{x_i}\phi_i + \frac{1}{2}\sum_{i=1}^{\ell}\sum_{j=1}^{\ell} f_{x_i,x_j}\nu_i'\nu_j + f\left[\frac{(1-\rho)r}{\rho} + \frac{(1-\rho)\lambda'\lambda}{2\rho^2}\right] + \frac{1-\rho}{\rho}\sum_{i=1}^{\ell} f_{x_i}\nu_i'\lambda = 0\,.$$

This can also be derived from the PDE (15.3) after making the correction to (15.3) stated in the errata: The rf on the right-hand side should be replaced by 0.

(c) Explain why the optimal portfolio is $\pi(t, X_t)$ where π satisfies

$$\pi(t,x)' = \left(\frac{1}{\rho}\lambda(x)' + \sum_{i=1}^{\ell}\frac{\partial \log f(t,x)}{\partial x_i}\nu_i(x)'\right)\sigma(x)^{-1}\,.$$

How does this compare to the optimal portfolio derived in Exercise 14.6 for a constant investment opportunity set? How does it compare to the optimal portfolio (14.24) derived from dynamic programming?

Solution: Wealth is $W_t = M_t^{-1/\rho} f(t, X_t)$, so the stochastic part of $\mathrm{d}W/W$ is

$$\frac{1}{f}\sum_{i=1}^{\ell} f_{x_i}\nu_i' \,\mathrm{d}B + \frac{1}{\rho}\lambda' \,\mathrm{d}B\,.$$

Choosing π so that this stochastic part equals $\pi'\sigma\,\mathrm{d}B$ yields the formula for π.

In Exercise 14.6, the optimal portfolio is

$$\pi = \frac{1}{\rho}\Sigma^{-1}(\mu - r\iota) = \frac{1}{\rho}(\sigma\sigma')^{-1}(\mu - r\iota)\,.$$

Using the fact that σ and σ' are invertible here and the fact that $\mu - r\iota = \sigma\lambda$, we can write this as

$$\pi = \frac{1}{\rho}(\sigma')^{-1}\sigma^{-1}\sigma\lambda = \frac{1}{\rho}(\sigma')^{-1}\lambda\,.$$

This agrees with the formula in this exercise when $f_{x_i} = 0$ for all i (as is true in Exercise 14.6).

Similarly, we the fact that σ and σ' are invertible here and the fact that $\mu - r\iota = \sigma\lambda$ to write (14.24) as

$$\pi = -\frac{J_w}{wJ_{ww}}(\sigma')^{-1}\lambda - \sum_{i=1}^{\ell}\frac{J_{wx_i}}{wJ_{ww}}(\sigma')^{-1}\nu_i\,.$$

This must agree with the formula for this exercise, so the relative risk aversion of the value function must be ρ and $J_{wx_i}/wJ_{ww} = f_{x_i}/f$ for all i.

15.4. Assume aggregate consumption C and its expected growth rate μ satisfy

$$\frac{\mathrm{d}C}{C} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_1$$

$$\mathrm{d}\mu = \kappa(\theta - \mu)\,\mathrm{d}t + \gamma\left[\rho\,\mathrm{d}B_1 + \sqrt{1-\rho^2}\,\mathrm{d}B_2\right]$$

for constants σ, κ, ρ, θ, and γ and independent Brownian motions B_1 and B_2. Then the vector process (C, μ) is Markov. Assume

$$M_t \stackrel{\text{def}}{=} \mathrm{e}^{-\delta t} \left(\frac{C_t}{C_0}\right)^{-\rho}$$

is an SDF process for constants δ and ρ. **The same symbol ρ is used here for the correlation between $\mathrm{d}C/C$ and $\mathrm{d}\mu$ and also for relative risk aversion. In the solution below, I use η instead of ρ for the correlation. That is, I assume**

$$\mathrm{d}\mu = \kappa(\theta - \mu)\,\mathrm{d}t + \gamma \left[\eta\,\mathrm{d}B_1 + \sqrt{1-\eta^2}\,\mathrm{d}B_2\right].$$

(a) Show that the price of risk vector λ for the SDF process M is constant.

Solution: The stochastic part of $\mathrm{d}M/M$ is $-\rho\sigma\,\mathrm{d}B_1$, so the price of risk vector is

$$\lambda = \begin{pmatrix} \rho\sigma_1 \\ 0 \end{pmatrix}.$$

(b) Explain why the market price-dividend ratio is a function of μ.

Solution: The market price-dividend ratio is given in part (c). Because μ is Markovian, μ_t is a sufficient statistic for forecasting C_u/C_t and hence also for forecasting M_u/M_t.

(c) Let $f(\mu)$ denote the market price-dividend ratio, that is,

$$f(\mu_t) = \mathsf{E}_t \int_t^\infty \frac{M_u}{M_t} \cdot \frac{C_u}{C_t}\,\mathrm{d}u\,.$$

Write down the fundamental ODE for f.

Solution: The price of the market portfolio is $C_t f(\mu_t)$, so the market return is

$$\frac{\mathrm{d}S}{S} \stackrel{\text{def}}{=} \frac{\mathrm{d}(Cf) + C\,\mathrm{d}t}{Cf} = \frac{\mathrm{d}C}{C} + \frac{\mathrm{d}f}{f} + \left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}f}{f}\right) + \frac{1}{f}\,\mathrm{d}t\,.$$

The market risk premium is

$$\mathsf{E}\left[\frac{\mathrm{d}S}{S}\right] - r\,\mathrm{d}t = \mathsf{E}\left[\frac{\mathrm{d}C}{C}\right] + \mathsf{E}\left[\frac{\mathrm{d}f}{f}\right] + \left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}f}{f}\right) + \frac{1}{f}\,\mathrm{d}t - r\,\mathrm{d}t\,.$$

The required risk premium is

$$
\begin{aligned}
-\left(\frac{\mathrm{d}S}{S}\right)\left(\frac{\mathrm{d}M}{M}\right) &= -\left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}M}{M}\right) - \left(\frac{\mathrm{d}f}{f}\right)\left(\frac{\mathrm{d}M}{M}\right) \\
&= \rho\left(\frac{\mathrm{d}C}{C}\right)^2 + \rho\left(\frac{\mathrm{d}f}{f}\right)\left(\frac{\mathrm{d}C}{C}\right).
\end{aligned}
$$

Equating the two gives

$$
\mu + \frac{f'}{f}\kappa(\theta-\mu) + \frac{1}{2}\frac{f''}{f}\gamma^2 + \frac{f'}{f}\sigma\gamma\eta + \frac{1}{f} - r = \rho\sigma^2 + \frac{f'}{f}\rho\sigma\gamma\eta\,.
$$

This is equivalent to

$$
[\kappa(\theta-\mu) + (1-\rho)\sigma\gamma\eta]f'(x) + \frac{1}{2}\gamma^2 f''(x) = (r + \rho\sigma^2 - \mu)f(x)\,.
$$

15.5. Assume two dividend processes D_i are independent geometric Brownian motions:

$$
\frac{\mathrm{d}D_i}{D_i} = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_i
$$

for constants μ_i and σ_i and independent Brownian motions B_i. Define $C_t = D_{1t} + D_{2t}$. Assume

$$
M_t \stackrel{\text{def}}{=} \mathrm{e}^{-\delta t}\frac{C_0}{C_t}
$$

is an SDF process. (This is the MRS for a log-utility investor.) Define $X_t = D_{1t}/C_t$.

(a) Show that X is a Markov process.

Solution: We have

$$
\begin{aligned}
\frac{\mathrm{d}C}{C} &= \frac{\mathrm{d}D_1}{C} + \frac{\mathrm{d}D_2}{C} \\
&= \frac{D_1(\mu_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_1)}{C} + \frac{D_2(\mu_2\,\mathrm{d}t + \sigma_2\,\mathrm{d}B_2)}{C} \\
&= X(\mu_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_1) + (1-X)(\mu_2\,\mathrm{d}t + \sigma_2\,\mathrm{d}B_2)\,.
\end{aligned}
$$

Using this result and the definition $X = D_1/C$, we obtain

$$\begin{aligned}\frac{\mathrm{d}X}{X} &= \frac{\mathrm{d}D_1}{D_1} - \frac{\mathrm{d}C}{C} - \left(\frac{\mathrm{d}D_1}{D_1}\right)\left(\frac{\mathrm{d}C}{C}\right) + \left(\frac{\mathrm{d}C}{C}\right)^2 \\ &= (1-X)(\mu_1\,\mathrm{d}t - \mu_2\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_1 - \sigma_2\,\mathrm{d}B_2) - X\sigma_1^2\,\mathrm{d}t + X^2\sigma_1^2\,\mathrm{d}t + (1-X)^2\sigma_2^2\,\mathrm{d}t\,.\end{aligned}$$

Thus, the distribution of $\mathrm{d}X$ is a function of X alone.

(b) Show that the value

$$\mathsf{E}_t \int_t^\infty \frac{M_u}{M_t} D_{1u}\,\mathrm{d}u$$

equals $C_t f(X_t)$ for some function f.

Solution: This is the value of the first asset. It equals

$$C_t\,\mathsf{E}_t \int_t^\infty \frac{M_u}{M_t} X_u\,\mathrm{d}u\,.$$

Because X is Markov and the growth rate $\mathrm{d}C/C$ depends only on X (as shown in the solution to part (a)) and because M is a function of C, X_t is a sufficient statistic for forecasting $M_u X_u/M_t$.

(c) Write down the ODE that f must satisfy.

Solution: As in the previous exercise, we can derive the ODE by equating the risk premium to the required risk premium. The rate of return of the first asset is

$$\frac{\mathrm{d}S}{S} \stackrel{\text{def}}{=} \frac{\mathrm{d}(Cf) + C\,\mathrm{d}t}{Cf} = \frac{\mathrm{d}C}{C} + \frac{\mathrm{d}f}{f} + \left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}f}{f}\right) + \frac{1}{f}\,\mathrm{d}t\,.$$

The risk premium is

$$\mathsf{E}\left[\frac{\mathrm{d}S}{S}\right] - r\,\mathrm{d}t = \mathsf{E}\left[\frac{\mathrm{d}C}{C}\right] + \mathsf{E}\left[\frac{\mathrm{d}f}{f}\right] + \left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}f}{f}\right) + \frac{1}{f}\,\mathrm{d}t - r\,\mathrm{d}t\,.$$

The required risk premium is

$$\begin{aligned} -\left(\frac{\mathrm{d}S}{S}\right)\left(\frac{\mathrm{d}M}{M}\right) &= -\left(\frac{\mathrm{d}C}{C}\right)\left(\frac{\mathrm{d}M}{M}\right) - \left(\frac{\mathrm{d}f}{f}\right)\left(\frac{\mathrm{d}M}{M}\right) \\ &= \left(\frac{\mathrm{d}C}{C}\right)^2 + \left(\frac{\mathrm{d}f}{f}\right)\left(\frac{\mathrm{d}C}{C}\right). \end{aligned}$$

When we equate the two, we can cancel the term

$$\left(\frac{\mathrm{d}f}{f}\right)\left(\frac{\mathrm{d}C}{C}\right)$$

that appears in both the risk premium and the required risk premium. Equating the two therefore gives

$$\begin{aligned} x\mu_1 + (1-x)\mu_2 + \frac{f'}{f}\left[x(1-x)(\mu_1-\mu_2) - x^2\sigma_1^2 + x^3\sigma_1^2 + x(1-x)^2\sigma_2^2\right] \\ + \frac{1}{2}\frac{f''}{f}\left[x^2(1-x)^2(\sigma_1^2+\sigma_2^2)\right] + \frac{1}{f} - r = x^2\sigma_1^2 + (1-x)^2\sigma_2^2\,. \end{aligned}$$

This is equivalent to

$$\begin{aligned} 1 + \left[x(1-x)(\mu_1-\mu_2) - x^2\sigma_1^2 + x^3\sigma_a^2 + x(1-x)^2\sigma_2^2\right]f'(x) + \frac{1}{2}x^2(1-x)^2(\sigma_1^2+\sigma_2^2)f''(x) \\ = \left[r - x\mu_1 - (1-x)\mu_2 + x^2\sigma_1^2 + (1-x)^2\sigma_2^2\right]f(x)\,. \end{aligned}$$

15.6. Adopt the notation of Exercise 13.3. Suppose M^dR^d is a martingale and define the risk neutral probability corresponding to M^d. Assume M^dXR^f is also a martingale. Show that

$$\frac{\mathrm{d}X}{X} = (r^d - r^f)\,\mathrm{d}t + \sigma_x\,\mathrm{d}B^*\,,$$

where B^* is a Brownian motion under the risk neutral probability. Note: This is called uncovered interest parity under the risk neutral probability. Suppose for example that $r^f < r^d$. Then it may appear profitable to borrow in the foreign currency and invest in the domestic currency money market. The result states that, under the risk neutral probability, the cost of the foreign currency is expected to increase so as to exactly offset the interest rate differential.

Solution: We could use Girsanov's theorem, or we can reason as follows. Because $M^d X R^f$ is a martingale under the physical probability, it follows that XR^f/R^d is a martingale under the risk neutral probability. Setting $Y = XR^f/R^d$, we have

$$\frac{\mathrm{d}Y}{Y} = (r^f - r^d)\,\mathrm{d}t + \frac{\mathrm{d}X}{X}\,.$$

This process has zero drift under the risk neutral probability, so the drift of $\mathrm{d}X/X$ under the risk neutral probability is $(r^d - r^f)\,\mathrm{d}t$. Because volatilities do not change when we make an equivalent change of measures, we know that

$$\frac{\mathrm{d}X}{X} = (r^d - r^f)\,\mathrm{d}t + \sigma_x\,\mathrm{d}B_x^*$$

for some Brownian motion B_x^* under the risk neutral probability. If fact, defining B_x^* by

$$\mathrm{d}B_x^* = \frac{1}{\sigma_x}\left(\frac{\mathrm{d}X}{X} - (r^d - r^f)\,\mathrm{d}t\right) = \frac{\mu_x - r^d + r^f}{\sigma_x}\,\mathrm{d}t + \mathrm{d}B_x\,,$$

the facts that B_x^* has no drift under the risk neutral probability and $(\mathrm{d}B_x^*)^2 = (\mathrm{d}B_x)^2 = \mathrm{d}t$ implies B_x^* is a Brownian motion under the risk neutral probability, by Levy's theorem.

15.7. Assume the market is complete, and let M denote the unique SDF process. Assume MR is a martingale. Consider $T < \infty$, and define the probability $\mathbb{Q}$ in terms of $\xi_T = M_T R_T$ by (15.5). Define B^* by (15.8). Let x be a random variable that depends only on the path of the vector process B^* up to time T and let C be a process adapted to B^*. Assume

$$\mathsf{E}^*\left[\int_0^T \frac{C_s}{R_s}\,\mathrm{d}s + \frac{x}{R_T}\right] < \infty\,.$$

For $t \leq T$, define

$$W_t^* = \mathsf{E}_t^*\left[\int_t^T \frac{C_s}{R_s}\,\mathrm{d}s + \frac{x}{R_T}\right]\,.$$

Set $W_t = R_t W_t^*$ (so, in particular, $W_T = x$). Use martingale representation under Q and the fact that

$$\int_0^t \frac{C_s}{R_s}\,\mathrm{d}s + W_t^* = \mathsf{E}_t^*\left[\int_0^T \frac{C_s}{R_s}\,\mathrm{d}s + \frac{x}{R_T}\right],$$

is a $\mathbb{Q}$–martingale to prove that there is a portfolio process ϕ such that W, C and ϕ satisfy the intertemporal budget constraint (13.38).

Solution: From (15.8), we have

$$\mathrm{d}B^* = \lambda\,\mathrm{d}t + \mathrm{d}B = \sigma^{-1}(\mu - r\iota)\,\mathrm{d}t + \mathrm{d}B\,.$$

By the martingale representation theorem, applied under the risk neutral probability, there exists ψ such that

$$\begin{aligned}\int_0^t \frac{C_s}{R_s}\,\mathrm{d}s + W_t^* &= W_0^* + \int_0^t \psi_s'\,\mathrm{d}B_s^* \\ &= W_0^* + \int_0^t \psi_s'\sigma_s^{-1}(\mu_s - r_s\iota)\,\mathrm{d}s + \int_0^t \psi_s'\,\mathrm{d}B_s\,.\end{aligned}$$

Set $\phi_s = R_s\left(\sigma_s^{-1}\right)'\psi_s$. Then, we have

$$\int_0^t \frac{C_s}{R_s}\,\mathrm{d}s + W_t^* = W_0^* + \int_0^t \frac{1}{R_s}\phi_s'(\mu_s - r_s\iota)\,\mathrm{d}s + \int_0^t \frac{1}{R_s}\phi_s'\sigma_s\,\mathrm{d}B_s\,.$$

Thus,

$$\frac{C}{R}\,\mathrm{d}t + \mathrm{d}W^* = \frac{1}{R}\phi'(\mu - r\iota)\,\mathrm{d}t + \frac{1}{R}\phi'\sigma\,\mathrm{d}B\,.$$

Because $W = RW^*$, we have

$$\begin{aligned}\mathrm{d}W &= R\,\mathrm{d}W^* + W^*\,\mathrm{d}R \\ &= R\left[\frac{1}{R}\phi'(\mu - r\iota)\,\mathrm{d}t + \frac{1}{R}\phi'\sigma\,\mathrm{d}B - \frac{C}{R}\,\mathrm{d}t\right] + W\frac{\mathrm{d}R}{R} \\ &= rW\,\mathrm{d}t + \phi'(\mu - r\iota)\,\mathrm{d}t + \phi'\sigma\,\mathrm{d}B\,.\end{aligned}$$

15.8. This exercise verifies that, as asserted in Section 15.3, condition (15.9) is sufficient for MW to be a martingale. Let M be an SDF process such that MR is a martingale. Define B^* by (15.8). Let W be a positive self-financing wealth process. Define $W^* = W/R$.

(a) Use Itô's formula, (13.10), (13.20), and (15.8) to show that

$$\mathrm{d}W^* = \frac{1}{R}\phi'\sigma\,\mathrm{d}B^*\,.$$

Solution: We have

$$\begin{aligned}\frac{\mathrm{d}W^*}{W^*} &= \frac{\mathrm{d}W}{W} - \frac{\mathrm{d}R}{R} \\ &= \pi'(\mu - r\iota)\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}B \\ &= \pi'\sigma\lambda\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}B \\ &= \pi'\sigma\,\mathrm{d}B^*\,,\end{aligned}$$

using Itô's formula, (13.10), (13.20) and (15.8) successively. This implies

$$\begin{aligned}\mathrm{d}W^* &= W^*\pi'\sigma\,\mathrm{d}B^* \\ &= \frac{W}{R}\pi'\sigma\,\mathrm{d}B^* \\ &= \frac{1}{R}\phi'\sigma\,\mathrm{d}B^*\,,\end{aligned}$$

where $\phi = W\pi$.

(b) Explain why the condition

$$\mathsf{E}^*\left[\int_0^T \frac{1}{R^2}\phi'\Sigma\phi\,\mathrm{d}t\right] < \infty$$

implies that W^* is a martingale on $[0,T]$ under the risk neutral probability defined from M, where E^* denotes expectation with respect to the risk neutral probability.