imply

$$\mathsf{E}_t\left[M_T W_T^\dagger \frac{C_s}{W_s^\dagger}\right] = \mathsf{E}_t\left[\frac{C_s}{W_s^\dagger}\mathsf{E}_s\left[M_T W_T^\dagger\right]\right]$$
$$= \mathsf{E}_t\left[M_s C_s\right] .$$

Therefore,

$$\begin{aligned}\mathsf{E}_t\left[M_T W_T^\dagger \int_0^T \frac{C_s}{W_s^\dagger}\,\mathrm{d}s\right] &= \int_0^t \frac{C_s}{W_s^\dagger}\,\mathrm{d}s \times \mathsf{E}_t\left[M_T W_T^\dagger\right] + \mathsf{E}_t\left[M_T W_T^\dagger \int_t^T \frac{C_s}{W_s^\dagger}\,\mathrm{d}s\right] \\ &= M_t W_t^\dagger \int_0^t \frac{C_s}{W_s^\dagger}\,\mathrm{d}s + \int_t^T \mathsf{E}_t\left[M_T W_T^\dagger \frac{C_s}{W_s^\dagger}\right]\mathrm{d}s \\ &= M_t W_t^\dagger \int_0^t \frac{C_s}{W_s^\dagger}\,\mathrm{d}s + \int_t^T \mathsf{E}_t\left[M_s C_s\right]\mathrm{d}s \\ &= M_t W_t^\dagger \int_0^t \frac{C_s}{W_s^\dagger}\,\mathrm{d}s + \mathsf{E}_t\left[\int_t^T M_s C_s\,\mathrm{d}s\right] .\end{aligned}$$

(d) Let M be an SDF process and assume $MW^\dagger$ is a martingale. Use the results of the previous two parts to show that (13.39) is a martingale.

Solution: Let

$$X_t = \mathsf{E}_t\left[M_T W_T^\dagger \int_0^T \frac{C_s}{W_s^\dagger}\,\mathrm{d}s\right] .$$

This is a martingale. Using Parts (c) and (b) successively, we have

$$\begin{aligned}X_t &= M_t W_t^\dagger \int_0^t \frac{C_s}{W_s^\dagger}\,\mathrm{d}s + \mathsf{E}_t\left[\int_t^T M_s C_s\,\mathrm{d}s\right] \\ &= M_t(W_t^\dagger - W_t) + \mathsf{E}_t\left[\int_t^T M_s C_s\,\mathrm{d}s\right] .\end{aligned}$$

From the assumption that $MW^\dagger$ is a martingale, it follows that

$$M_t W_t - \mathsf{E}_t\left[\int_t^T M_s C_s\,\mathrm{d}s\right]$$

is a martingale. The second term in this expression is zero at $t = T$. Therefore,

$$M_t W_t - \mathsf{E}_t\left[\int_t^T M_s C_s\,\mathrm{d}s\right] = \mathsf{E}_t[M_T W_T] .$$

This implies

$$\begin{aligned}
M_tW_t + \int_0^t M_sC_s\,\mathrm{d}s &= \int_0^t M_sC_s\,\mathrm{d}s + \mathsf{E}_t\left[\int_t^T M_sC_s\,\mathrm{d}s\right] + \mathsf{E}_t[M_TW_T] \\
&= \mathsf{E}_t\left[M_TW_T + \int_0^T M_sC_s\,\mathrm{d}s\right] .
\end{aligned}$$

13.6. Suppose W, C and π satisfy the intertemporal budget constraint (13.38) Define

$$W_t^\dagger = W_t + R_t\int_0^t \frac{C_s}{R_s}\,\mathrm{d}s\,.$$

Note: This means consumption is reinvested in the money market account rather than in the portfolio generating the wealth process as in (13.43).

(a) Show that $W^\dagger$ satisfies the intertemporal budget constraint (13.12).

Solution: We have

$$\begin{aligned}
\mathrm{d}W^\dagger &= \mathrm{d}W + C\,\mathrm{d}t + \left(\int_0^t \frac{C_s}{R_s}\,\mathrm{d}s\right)\mathrm{d}R \\
&= \mathrm{d}W + C\,\mathrm{d}t + (W^\dagger - W)\frac{\mathrm{d}R}{R} \\
&= rW\,\mathrm{d}t + \phi'(\mu - r\iota)\,\mathrm{d}t + \phi'\sigma\,\mathrm{d}B + (W^\dagger - W)r\,\mathrm{d}t \\
&= rW^\dagger\,\mathrm{d}t + \phi'(\mu - r\iota)\,\mathrm{d}t + \phi'\sigma\,\mathrm{d}B\,.
\end{aligned}$$

(b) Let M be an SDF process. Assume MR is a martingale and $MW^\dagger$ is a martingale. Deduce that (13.39) is a martingale.

Solution: From the definition of $W^\dagger$, we have

$$\int_0^t M_sC_s\,\mathrm{d}s + M_tW_t = \int_0^t M_sC_s\,\mathrm{d}s + M_tW_t^\dagger - M_tR_t\int_0^t \frac{C_s}{R_s}\,\mathrm{d}s\,.$$

Given the assumption that $MW^\dagger$ is a martingale, it suffices to show that

$$\int_0^t M_sC_s\,\mathrm{d}s - M_tR_t\int_0^t \frac{C_s}{R_s}\,\mathrm{d}s$$

is a martingale. By iterated expectations and the assumption that MR is a martingale, we obtain, using the same reasoning as in the previous exercise,

$$\begin{aligned}\mathsf{E}_t\left[M_T R_T \int_0^T \frac{C_s}{R_s}\,\mathrm{d}s\right] &= \int_0^t \frac{C_s}{R_s}\,\mathrm{d}s \times \mathsf{E}_t[M_T R_T] + \mathsf{E}_t\left[M_T R_T \int_t^T \frac{C_s}{R_s}\,\mathrm{d}s\right] \\ &= M_t R_t \int_0^t \frac{C_s}{R_s}\,\mathrm{d}s + \mathsf{E}_t\left[\int_t^T M_s C_s\,\mathrm{d}s\right].\end{aligned}$$

Thus,

$$\begin{aligned}\mathsf{E}_t\left[\int_0^T M_s C_s\,\mathrm{d}s - M_T R_T \int_0^T \frac{C_s}{R_s}\,\mathrm{d}s\right] &= \mathsf{E}_t\left[\int_0^T M_s C_s\,\mathrm{d}s\right] - M_t R_t \int_0^t \frac{C_s}{R_s}\,\mathrm{d}s - \mathsf{E}_t\left[\int_t^T M_s C_s\,\mathrm{d}s\right] \\ &= \int_0^t M_s C_s\,\mathrm{d}s - M_t R_t \int_0^t \frac{C_s}{R_s}\,\mathrm{d}s\end{aligned}$$

Chapter 14

Continuous-Time Portfolio Choice and Pricing

14.1. Assume the continuous-time CAPM holds:

$$(\mu_i - r)\,\mathrm{d}t = \rho \left(\frac{\mathrm{d}S_i}{S_i}\right)\left(\frac{\mathrm{d}W_m}{W_m}\right)$$

for each asset i, where W_m denotes the value of the market portfolio, $\rho = \alpha W_m$, and α denotes the aggregate absolute risk aversion. Define $\sigma_i = \sqrt{e_i' \Sigma e_i}$ to be the volatility of asset i, as described in Section 13.1, so we have

$$\frac{\mathrm{d}S_i}{S_i} = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}Z_i$$

for a Brownian motion Z_i. Likewise, the return on the market portfolio is

$$\frac{\mathrm{d}W_m}{W_m} = \mu_m\,\mathrm{d}t + \sigma_m\,\mathrm{d}Z_m$$

for some μ_m, σ_m and Brownian motion Z_m. Let ϕ_{im} denote the correlation process of the Brownian motions Z_i and Z_m.

(a) Using the fact that the market return must also satisfy the continuous-time CAPM, show that the continuous-time CAPM can be written as

$$\mu_i - r = \frac{\sigma_i \sigma_m \phi_{im}}{\sigma_m^2}(\mu_m - r)\,.$$

Solution: We have

$$(\mu_m - r)\,\mathrm{d}t = \rho\left(\frac{\mathrm{d}W_m}{W_m}\right)^2 = \rho\sigma_m^2\,\mathrm{d}t\,,$$

so $\rho = (\mu_m - r)/\sigma_m^2$. Therefore

$$\begin{aligned}(\mu_i - r)\,\mathrm{d}t &= \frac{\mu_m - r}{\sigma_m^2}\left(\frac{\mathrm{d}S_i}{S_i}\right)\left(\frac{\mathrm{d}W_m}{W_m}\right)\\ &= \frac{\mu_m - r}{\sigma_m^2}(\sigma_i\sigma_m\phi_{im})\,\mathrm{d}t\,.\end{aligned}$$

(b) Suppose r, μ_i, μ_m, σ_i, σ_m and ρ_i are constant over a time interval Δt, so both S_i and W_m are geometric Brownian motions over the time interval. Define the annualized continuously compounded rates of return over the time interval:

$$r_i = \frac{\Delta \log S_i}{\Delta t} \qquad \text{and} \qquad r_m = \frac{\Delta \log W_m}{\Delta t}\,.$$

Let $\overline{r}_i$ and $\overline{r}_m$ denote the expected values of r_i and r_m. Show that the continuous-time CAPM implies

$$\overline{r}_i - r = \frac{\operatorname{cov}(r_i, r_m)}{\operatorname{var}(r_m)}(\overline{r}_m - r) + \frac{1}{2}[\operatorname{cov}(r_i, r_m) - \operatorname{var}(r_i)]\,\Delta t\,.$$

Solution: We have $\mathsf{E}[\varDelta \log S_i] = (\mu_i - \sigma_i^2/2)\varDelta t$ and $\mathsf{E}[\varDelta \log S_m] = (\mu_m - \sigma_m^2/2)\varDelta t$, so

$$\bar{r}_i = \mu_i - \frac{1}{2}\sigma_i^2\,, \qquad \text{and} \qquad \bar{r}_m = \mu_m - \frac{1}{2}\sigma_m^2\,.$$

From Part (a),

$$
\begin{aligned}
\bar{r}_i - r &= \mu_i - r - \frac{1}{2}\sigma_i^2 \\
&= \frac{\sigma_i\sigma_m\phi_{im}}{\sigma_m^2}(\mu_m - r) - \frac{1}{2}\sigma_i^2 \\
&= \frac{\sigma_i\sigma_m\phi_{im}}{\sigma_m^2}\left(\bar{r}_m + \frac{1}{2}\sigma_m^2 - r\right) - \frac{1}{2}\sigma_i^2 \\
&= \frac{\sigma_i\sigma_m\phi_{im}}{\sigma_m^2}\left(\bar{r}_m - r\right) + \frac{1}{2}\sigma_i\sigma_m\phi_{im} - \frac{1}{2}\sigma_i^2 \,.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathrm{var}(r_i) &= \frac{\sigma_i^2}{\Delta t}\,, \\
\mathrm{var}(r_m) &= \frac{\sigma_m^2}{\Delta t}\,, \\
\mathrm{cov}(r_i, r_m) &= \frac{\sigma_i\sigma_m\phi_{im}}{\Delta t}\,.
\end{aligned}
$$

Making these substitutions yields the result.

14.2. For each investor $h = 1, \ldots, H$, let π_h denote the optimal portfolio presented in (14.24). Using the notation of Section 14.6, set $\tau_h = 1/\alpha_h$ for each investor h. Then, (14.24) implies

$$
W_h\pi_h = \tau_h\Sigma^{-1}(\mu - r\iota) - \sum_{j=1}^{\ell} \tau_h \frac{\eta_{hj}}{X_j}\Sigma^{-1}\sigma\nu_j \,.
$$

The formula given in the exercise is inconsistent with the notation in the chapter. η_{hj} **should be divided by** X_j **as here.**

(a) Deduce that

$$
\mu - r\iota = \alpha W \Sigma \pi + \sum_{j=1}^{\ell} \frac{\eta_j}{X_j}\sigma\nu_j \,, \tag{14.33}
$$

(this formula is also incorrectly stated in the exercise. η_j **should be divided by** X_j **as here)** where π denotes the market portfolio:

$$
\pi = \sum_{h=1}^{H} \frac{W_h}{W}\pi_h \,.
$$

Solution: We have

$$\tau_h(\mu - r\iota) = W_h\Sigma\pi_h + \sum_{j=1}^{\ell} \tau_h \frac{\eta_{hj}}{X_j}\sigma\nu_j \,.$$

Summing over h yields

$$\tau(\mu - r\iota) = \Sigma \sum_{h=1}^{H} W_h\pi_h + \sum_{j=1}^{\ell}\left(\sum_{h=1}^{H}\tau_h\frac{\eta_{hj}}{X_j}\right)\sigma\nu_j \,,$$

which implies

$$\mu - r\iota = \alpha W\Sigma\pi + \sum_{j=1}^{\ell}\frac{1}{X_j}\left(\sum_{h=1}^{H}\frac{\tau_h\eta_{hj}}{\tau}\right)\sigma\nu_j \,.$$

The result follows from the definition

$$\eta_j = \sum_{h=1}^{H}\frac{\tau_h\eta_{hj}}{\tau} \,.$$

(b) Explain why (14.33) is the same as the ICAPM (14.31).

Solution: Stacking the equations (14.31a) for $i = 1, \ldots, n$ yields

$$\begin{aligned}
(\mu - r\iota)\,\mathrm{d}t &= \rho(\sigma\,\mathrm{d}B)\left(\frac{\mathrm{d}W}{W}\right) + \sum_{j=1}^{\ell}\eta_j(\sigma\,\mathrm{d}B)\left(\frac{\mathrm{d}X_j}{X_j}\right) \\
&= \alpha W(\sigma\,\mathrm{d}B)(\mathrm{d}B')\sigma'\pi + \sum_{j=1}^{\ell}\frac{\eta_j}{X_j}(\sigma\,\mathrm{d}B)(\mathrm{d}B')\nu_j \\
&= \alpha W\Sigma\pi\,\mathrm{d}t + \sum_{j=1}^{\ell}\frac{\eta_j}{X_j}\sigma\nu_j\,\mathrm{d}t \,.
\end{aligned}$$

14.3. Consider an investor with initial wealth $W_0 > 0$ who seeks to maximize $\mathsf{E}[\log W_T]$. Assume

$$\mathsf{E}\left[\int_0^T |r_t|\,\mathrm{d}t\right] < \infty \qquad \text{and} \qquad \mathsf{E}\left[\int_0^T \kappa_t^2\,\mathrm{d}t\right] < \infty \,,$$

where κ denotes the maximum Sharpe ratio. Assume portfolio processes are constrained to satisfy

$$\mathsf{E}\left[\int_0^T \pi_t'\Sigma_t\pi_t\,\mathrm{d}t\right] < \infty \,.$$

Recall that this constraint implies

$$\mathsf{E}\left[\int_0^T \pi'\sigma\,\mathrm{d}B\right] = 0\,.$$

(a) Using the formula (13.15) for W_t show that the optimal portfolio process is

$$\pi = \Sigma^{-1}(\mu - r\iota)\,.$$

Hint: the objective function obtained by substituting the formula (13.15) for W_t can be maximized in π separately at each date and in each state of the world.

Solution: From (13.15), the realized utility is

$$\log W_0 + \int_0^T \left(r_s + \pi_s'(\mu_s - r_s\iota) - \frac{1}{2}\pi_s'\Sigma_s\pi_s\right)\,\mathrm{d}s + \int_0^T \pi_s'\sigma_s\,\mathrm{d}B_s\,.$$

The assumption

$$\mathsf{E}\left[\int_0^T \pi_s'\Sigma_s\pi_s\,\mathrm{d}s\right] < \infty\,,$$

implies

$$\int_0^t \pi_s'\sigma_s\,\mathrm{d}B_s$$

is a martingale. Thus, the expected utility is

$$\log W_0 + \mathsf{E}\left[\int_0^T \left(r_s + \pi_s'(\mu_s - r_s\iota) - \frac{1}{2}\pi_s'\Sigma_s\pi_s\right)\,\mathrm{d}s\right]\,.$$

This is maximized by maximizing

$$\pi_s'(\mu_s - r_s\iota) - \frac{1}{2}\pi_s'\Sigma_s\pi_s$$

for each s, implying

$$\pi_s = \Sigma_s^{-1}(\mu_s - r_s\iota)\,.$$

(b) Assume the market is Markovian. Show that the investor's value function is $V(t,w,x) = \log w + f(t,x)$, where

$$f(t,x) = \mathsf{E}\left[\int_t^T \left(r_s + \frac{1}{2}\kappa_s^2\right) \mathrm{d}s \,\middle|\, X_t = x\right] .$$

Solution: Repeating the above argument starting at date t instead of date 0 shows that the expected utility at t is

$$\log W_t + \mathsf{E}_t\left[\int_t^T \left(r_s + \pi_s'(\mu_s - r_s\iota) - \frac{1}{2}\pi_s'\Sigma_s\pi_s\right) \mathrm{d}s\right] .$$

Substituting the optimum $\pi_s = \Sigma_s^{-1}(\mu_s - r_s\iota)$ and recalling that $\kappa^2 = (\mu - r\iota)'\Sigma^{-1}(\mu - r\iota)$ yields the result.

14.4. Consider an investor with log utility and an infinite horizon. Assume the capital market line is constant, so we can write $J(w)$ instead of $J(x,w)$ for the value function.

(a) Show that

$$J(w) = \frac{\log w}{\delta} + K$$

solves the HJB equation (14.25), where

$$K = \frac{\log \delta}{\delta} + \frac{r - \delta + \kappa^2/2}{\delta^2} .$$

Show that $c = \delta w$ and $\pi = \Sigma^{-1}(\mu - r\iota)$ achieve the maximum in the HJB equation.

Solution: Substituting $J = K + \log w/\delta$, $J_w = 1/(\delta w)$ and $J_{ww} = -1/(\delta w^2)$, the HJB equation (14.25) is

$$0 = \max_{c,\pi}\left\{\log c - K\delta - \log w + \frac{1}{\delta}\left[r + \pi'(\mu - r\iota) - \frac{c}{w}\right] - \frac{1}{2\delta}\pi'\Sigma\pi\right\} .$$

The maximum is achieved at $c = \delta w$ and $\pi = \Sigma^{-1}(\mu - r\iota)$. Substituting these into the HJB equation, it reduces to the formula given for K.

(b) Show that the transversality condition

$$\lim_{T\to\infty} \mathsf{E}\left[\mathrm{e}^{-\delta T} J(W_T^*)\right] = 0$$

holds, where W^* denotes the wealth process generated by the consumption and portfolio processes in part (a).

Solution: We have

$$\begin{aligned}\frac{\mathrm{d}W^*}{W^*} &= \left(r + \pi'(\mu - r) - \frac{C^*}{W^*}\right) \mathrm{d}t + \pi'\sigma\,\mathrm{d}B \\ &= (r + \kappa^2 - \delta)\,\mathrm{d}t + (\mu - r)'\Sigma^{-1}\sigma\,\mathrm{d}B\,.\end{aligned}$$

Hence,

$$\mathrm{d}\log W^* = \left(r - \delta + \frac{1}{2}\kappa^2\right)\,\mathrm{d}t + (\mu - r)'\Sigma^{-1}\sigma\,\mathrm{d}B\,.$$

This implies that

$$\mathsf{E}\left[\mathrm{e}^{-\delta t}\log W_T^*\right] = \mathrm{e}^{-\delta T}\log W_0 + \mathrm{e}^{-\delta T}\left(r - \delta + \frac{1}{2}\kappa^2\right) T \to 0$$

as $T \to \infty$.

14.5. Consider an investor with power utility and an infinite horizon. Assume the capital market line is constant, so we can write $J(w)$ instead of $J(x, w)$ for the value function.

(a) Define

$$\xi = \frac{\delta - (1-\rho)r}{\rho} - \frac{(1-\rho)\kappa^2}{2\rho^2}\,.$$

Assume (14.26) holds, so $\xi > 0$. Show that

$$J(w) = \xi^{-\rho}\left(\frac{1}{1-\rho}w^{1-\rho}\right)$$

solves the HJB equation (14.25). Show that $c = \xi w$ and $\pi = (1/\rho)\Sigma^{-1}(\mu - r\iota)$ achieve the maximum in the HJB equation.

Solution: Substituting $J = \xi^{-\rho}w^{1-\rho}/(1-\rho)$, $wJ_w = \xi^{-\rho}w^{1-\rho}$ and $w^2 J_{ww} = -\rho\xi^{-\rho}w^{1-\rho}$, the HJB equation (14.25) is

$$0 = \max_{c,\pi} \left\{ \frac{1}{1-\rho}c^{1-\rho} - \frac{\delta}{1-\rho}\xi^{-\rho}w^{1-\rho} + \left[r + \pi'(\mu - r\iota) - \frac{c}{w}\right]\xi^{-\rho}w^{1-\rho} - \frac{1}{2}\rho\xi^{-\rho}w^{1-\rho}\pi'\Sigma\pi \right\} .$$

The maximum is achieved at $c = \xi w$ and $\pi = (1/\rho)\Sigma^{-1}(\mu - r\iota)$. Substituting these into the HJB equation, it reduces to the formula given for ξ.

(b) Show that, under the assumption $\xi > 0$, the transversality condition

$$\lim_{T\to\infty} \mathsf{E}\left[\mathrm{e}^{-\delta t} J(W_T^*)\right] = 0$$

holds, where W^* denotes the wealth process generated by the consumption and portfolio processes in part (a).

Solution: We have

$$\begin{aligned}
\frac{\mathrm{d}W^*}{W^*} &= \left(r + \pi'(\mu - r) - \frac{C^*}{W^*}\right)\mathrm{d}t + \pi'\sigma\,\mathrm{d}B \\
&= \left(r + \frac{\kappa^2}{\rho} - \xi\right)\mathrm{d}t + \frac{1}{\rho}(\mu - r)'\Sigma^{-1}\sigma\,\mathrm{d}B\,.
\end{aligned}$$

Hence,

$$W_T^* = W_0 \exp\left(\left(r - \xi + \frac{\kappa^2}{\rho} - \frac{\kappa^2}{2\rho^2}\right)T + \frac{1}{\rho}\int_0^T (\mu - r)'\Sigma^{-1}\sigma\,\mathrm{d}B_t\right) .$$

This implies

$$(W_T^*)^{1-\rho} = W_0^{1-\rho} \exp\left((1-\rho)\left(r - \xi + \frac{\kappa^2}{\rho} - \frac{\kappa^2}{2\rho^2}\right)T + \frac{1-\rho}{\rho}\int_0^T (\mu - r)'\Sigma^{-1}\sigma\,\mathrm{d}B_t\right) .$$

Thus,

$$\begin{aligned}
\mathsf{E}\left[\mathrm{e}^{-\delta T}(W_T^*)^{1-\rho}\right] &= W_0^{1-\rho}\exp\left(\left\{-\delta + (1-\rho)\left(r - \xi + \frac{\kappa^2}{\rho} - \frac{\kappa^2}{2\rho^2}\right) + \frac{(1-\rho)^2\kappa^2}{2\rho^2}\right\}T\right) \\
&= W_0^{1-\rho}\exp\left(\left\{-\delta + (1-\rho)(r - \xi) + \frac{(1-\rho)\kappa^2}{2\rho}\right\}T\right) .
\end{aligned}$$

The transversality condition holds if and only if

$$-\delta + (1-\rho)(r-\xi) + \frac{(1-\rho)\kappa^2}{2\rho} < 0\,.$$

Substituting the formula for ξ into this and rearranging shows that it is equivalent to

$$\frac{\delta-(1-\rho)r}{\rho} - \frac{(1-\rho)\kappa^2}{2\rho^2} > 0\,.$$

14.6. Consider an investor with power utility and a finite horizon. Assume the capital market line is constant and the investor is constrained to always have nonnegative wealth. Let $M = M_p$. Calculate the optimal portfolio as follows.

(a) Using (14.12), show that, for $s > t$,

$$\mathsf{E}_t\left[M_s^{1-1/\rho}\right] = M_t^{1-1/\rho}\mathrm{e}^{\alpha(s-t)}\,,$$

for a constant α.

Solution: For $s \geq t$, we have

$$M_s = M_t \exp\left(-r(s-t) - \frac{1}{2}\kappa^2(s-t) - \kappa(Z_s - Z_t)\right)\,.$$

Therefore,

$$M_s^{1-1/\rho} = M_t^{1-1/\rho}\exp\left(-r(s-t)(\rho-1)/\rho - \frac{1}{2}\kappa^2(s-t)(\rho-1)/\rho - \kappa(Z_s-Z_t)(\rho-1)/\rho\right)\,.$$

The exponential is of a normally distributed variable, so

$$\mathsf{E}_t\left[M_s^{1-1/\rho}\right] = M_t^{1-1/\rho}\mathrm{e}^{\alpha(s-t)}\,,$$

where

$$\alpha = -r(\rho-1)/\rho - \frac{1}{2}\kappa^2(\rho-1)/\rho + \frac{1}{2}\kappa^2(\rho-1)^2/\rho^2\,.$$

(b) Define C_t and W_T from the first-order conditions (14.7) and set

$$W_t = \mathsf{E}_t\left[\int_t^T \frac{M_s}{M_t} C_s \,\mathrm{d}s + \frac{M_T}{M_t} W_T\right].$$

Show that

$$W_t = g(t) M_t^{-1/\rho}$$

for some deterministic function g (which you could calculate).

Solution: The first order conditions are

$$\mathrm{e}^{-\delta t} C_t^{-\rho} = \gamma M_t \quad \Rightarrow \quad C_t = \left(\mathrm{e}^{\delta t} \gamma M_t\right)^{-1/\rho},$$

$$\beta W_T^{-\rho} = \gamma M_T \quad \Rightarrow \quad W_T = (\gamma M_T/\beta)^{-1/\rho}.$$

Thus,

$$\begin{aligned}
W_t &= \frac{1}{M_t}\mathsf{E}_t\left[\int_t^T M_s C_s \,\mathrm{d}s + M_T W_T\right] \\
&= \frac{1}{M_t}\mathsf{E}_t\left[\int_t^T \left(\gamma \mathrm{e}^{\delta s}\right)^{-1/\rho} M_s^{1-1/\rho} \,\mathrm{d}s + (\gamma/\beta)^{-1/\rho} M_T^{1-1/\rho}\right] \\
&= \frac{1}{M_t}\int_t^T \left(\gamma \mathrm{e}^{\delta s}\right)^{-1/\rho} \mathsf{E}_t\left[M_s^{1-1/\rho}\right] \mathrm{d}s + \frac{1}{M_t}(\gamma/\beta)^{-1/\rho}\mathsf{E}_t\left[M_T^{1-1/\rho}\right] \\
&= M_t^{-1/\rho}\int_t^T \left(\gamma \mathrm{e}^{\delta s}\right)^{-1/\rho} \mathrm{e}^{\alpha(s-t)} \,\mathrm{d}s + (\gamma/\beta)^{-1/\rho} M_t^{-1/\rho} \mathrm{e}^{\alpha(T-t)}.
\end{aligned}$$

Hence,

$$W_t = M_t^{-1/\rho} g(t),$$

where

$$g(t) = \int_t^T \left(\gamma \mathrm{e}^{\delta s}\right)^{-1/\rho} \mathrm{e}^{\alpha(s-t)} \,\mathrm{d}s + (\gamma/\beta)^{-1/\rho} \mathrm{e}^{\alpha(T-t)}.$$

(c) By applying Itô's formula to W in Part (b), show that the optimal portfolio is

$$\frac{1}{\rho}\Sigma^{-1}(\mu - r\iota).$$