in the two cases. It is easy to compare these numerically for various parameter values and to see that the value is larger in case (a) than in case (b), so the log equity premium is larger in case (a) than in case (b). Because of risk aversion, the possibility of a loss distributed uniformly over $[0.b^*]$ is feared more than a loss of $b^*/2$, so investors react more, being less willing to hold risky assets. This produces the larger log equity premium in case (a).

11.3. Let C denote aggregate consumption, and assume consumption growth C_{t+1}/C_t is IID. Assume

$$\frac{M_{t+1}}{M_t} \stackrel{\text{def}}{=} \delta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} + \alpha \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

is an SDF process for some δ, ρ, α, and γ. For $\alpha > 0$, condition (11.62a) of the Constantinides-Duffie model is satisfied. Take $\delta = 0.99$ and $\rho = 10$ as in Mehra and Prescott (2003).

(a) Apply the Gordon growth model (Section 10.4) to derive formulas for the risk-free return and expected market return.

Solution: The expected market return is

$$\frac{\mathsf{E}[C_{t+1}/C_t]}{\delta\mathsf{E}[(M_{t+1}/M_t)(C_{t+1}/C_t)]} = \frac{\mathsf{E}[C_{t+1}/C_t]}{\delta\mathsf{E}[(C_{t+1}/C_t)^{1-\rho}] + \delta\alpha\mathsf{E}[(C_{t+1}/C_t)^{1-\gamma}]}\,.$$

The risk-free return is

$$\frac{1}{\delta\mathsf{E}[M_{t+1}/M_t]} = \frac{1}{\mathsf{E}[(C_{t+1}/C_t)^{-\rho}] + \alpha\mathsf{E}[(C_{t+1}/C_t)^{-\gamma}]}\,.$$

(b) Explain why it is possible to choose parameters α and γ such that the risk-free return is as low as desired.

Solution: The denominator of the risk-free return converges to infinity as $\alpha \to \infty$, so the risk-free return converges to zero as $\alpha \to \infty$.

(c) Are there limits as to how high you can make the equity premium? Explain.

Solution: We'll work with the ratio $\mathsf{E}[R_m]/R_f$. From the previous part,

$$\frac{\mathsf{E}[R_m]}{R_f} = \mathsf{E}[C_{t+1}/C_t] \cdot \frac{\mathsf{E}[(C_{t+1}/C_t)^{-\rho}] + \alpha\mathsf{E}[(C_{t+1}/C_t)^{-\gamma}]}{\delta\mathsf{E}[(C_{t+1}/C_t)^{1-\rho}] + \delta\alpha\mathsf{E}[(C_{t+1}/C_t)^{1-\gamma}]}\,.$$

As $\alpha \to 0$, this approaches the same ratio in the CRRA model. The ratio is an increasing function of α if

$$\mathsf{E}[(C_{t+1}/C_t)^{1-\rho}] \cdot \mathsf{E}[(C_{t+1}/C_t)^{-\gamma}] > \mathsf{E}[(C_{t+1}/C_t)^{1-\gamma}] \cdot \mathsf{E}[(C_{t+1}/C_t)^{-\rho}]\,.$$

Assuming lognormality, this is equivalent to $\gamma > \rho$. As $\alpha \to \infty$, we have

$$\frac{\mathsf{E}[R_m]}{R_f} \to \mathsf{E}[C_{t+1}/C_t] \cdot \frac{\mathsf{E}[(C_{t+1}/C_t)^{-\gamma}]}{\delta\mathsf{E}[(C_{t+1}/C_t)^{1-\gamma}]}\,.$$

This is the same as in the CRRA model with $\rho = \gamma$. Assuming lognormality, we have

$$\frac{\mathsf{E}[R_m]}{R_f} \to \mathrm{e}^{\gamma\sigma^2}\,.$$

We can make this as large as we like by taking γ large.

11.4. Consider consumption processes (ii) and (iii) in Section 11.6. Take $T = 2$. Suppose consumption C_0 is known at date 0 (before any coins are tossed). Assume the power certainty equivalent and the CES aggregator.

(a) Assume two coins are tossed at date 0 determining C_1 and C_2. Calculate the utility U_0 of the person before the coins are tossed.

Solution: Because C_2 is known at date 1, we have $\xi_1 = C_2$. Therefore,

$$U_1 = V(C_1, C_2) = \left(C_1^{1-\alpha} + \delta C_2^{1-\alpha}\right)^{\frac{1}{1-\alpha}}\,.$$

This implies

$$\xi_0^{1-\rho} = \mathsf{E}[U_1^{1-\rho}] = \frac{1}{4}\sum_{x\in\{a,b\}}\sum_{y\in\{a,b\}}\left(x^{1-\alpha} + \delta y^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}}\,.$$

It follows that

$$U_0 = V(C_0, \xi_0) = \left\{ C_0^{1-\alpha} + \delta \left[\frac{1}{4} \sum_{x\in\{a,b\}} \sum_{y\in\{a,b\}} \left(x^{1-\alpha} + \delta y^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}} \right]^{\frac{1-\alpha}{1-\rho}} \right\}^{\frac{1}{1-\alpha}}.$$

(b) Assume a coin is tossed at date 1 determining C_1, and a coin is tossed at date 2 determining C_2. Calculate the utility U_0.

Solution: We have

$$\xi_1^{1-\rho} = \mathsf{E}_1[U_2^{1-\rho}] = \frac{1}{2} \sum_{y\in\{a,b\}} y^{1-\rho}.$$

Therefore,

$$U_1 = V(C_1, \xi_1) = \left[C_1^{1-\alpha} + \delta \left(\frac{1}{2} \sum_{y\in\{a,b\}} y^{1-\rho} \right)^{\frac{1-\alpha}{1-\rho}} \right]^{\frac{1}{1-\alpha}}.$$

Hence,

$$\xi_0^{1-\rho} = \mathsf{E}[U_1^{1-\rho}] = \frac{1}{2} \sum_{x\in\{a,b\}} \left[x^{1-\alpha} + \delta \left(\frac{1}{2} \sum_{y\in\{a,b\}} y^{1-\rho} \right)^{\frac{1-\alpha}{1-\rho}} \right]^{\frac{1-\rho}{1-\alpha}}.$$

It follows that

$$U_0 = V(C_0, \xi_0) = \left(C_0^{1-\alpha} + \delta \left\{ \frac{1}{2} \sum_{x\in\{a,b\}} \left[x^{1-\alpha} + \delta \left(\frac{1}{2} \sum_{y\in\{a,b\}} y^{1-\rho} \right)^{\frac{1-\alpha}{1-\rho}} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1-\alpha}{1-\rho}} \right)^{\frac{1}{1-\alpha}}.$$

(c) Show numerically that the utility is higher in part (a) than in part (b)—that is, early resolution of uncertainty is preferred—if $\rho > \alpha$. Show that late resolution is preferred if $\rho < \alpha$, and show that the person is indifferent about the timing of resolution of uncertainty if $\rho = \alpha$.

Solution: For example, take $a = 100$, $b = 200$, $C_0 = 150$, $\delta = 0.9$, and $\alpha = 4$. The utility in part (a) is

$$\rho = 2 \Rightarrow U_0 = 95.95\,, \quad \rho = 4 \Rightarrow U_0 = 92.63\,, \quad \rho = 6 \Rightarrow U_0 = 90.37\,.$$

The utility in part (b) is

$$\rho = 2 \Rightarrow U_0 = 97.48\,, \quad \rho = 4 \Rightarrow U_0 = 92.63\,, \quad \rho = 6 \Rightarrow U_0 = 89.48\,.$$

Chapter 12

Brownian Motion and Stochastic Calculus

12.1. Simulate the path of a Brownian motion over a year (using your favorite programming language or Excel) by simulating N standard normal random variables z_i and calculating $B_{t_i} = B_{t_{i-1}} + z_i\sqrt{\Delta t}$ for $i = 1, \ldots, N$, where $\Delta t = 1/N$ and $B_0 = 0$. (To simulate a standard normal random variable in a cell of an Excel worksheet, use the formula `= NORMSINV(RAND())`.)

(a) Plot the path—the set of points (t_i, B_{t_i}).

(b) Calculate the sum of the $(\Delta B_{t_i})^2$. Confirm that for large N the sum is approximately equal to 1.

(c) Calculate the sum of $|\Delta B_{t_i}|$. Confirm that this sum increases as N increases. Note: the sum converges to ∞ as $N \to \infty$ (because a Brownian motion has infinite total variation) but this may be difficult to see.

(d) Use the simulated Brownian motion to simulate a path of a geometric Brownian motion via the formula (12.23). Plot the path.

12.2. Assume X is an Itô process. Use Itô's formula to derive the following.

(a) Define $Y_t = \mathrm{e}^{X_t}$. Show that

$$\frac{\mathrm{d}Y}{Y} = \mathrm{d}X + \frac{1}{2}(\mathrm{d}X)^2 .$$

Solution: For $y = f(x) = \mathrm{e}^x$, we have $f'(x) = \mathrm{e}^x = y$ and $f''(x) = \mathrm{e}^x = y$, so

$$\begin{aligned} \mathrm{d}Y &= f'(X)\,\mathrm{d}X + \frac{1}{2}f''(X)(\mathrm{d}X)^2 \\ &= Y\,\mathrm{d}X + \frac{1}{2}Y(\mathrm{d}X)^2 . \end{aligned}$$

(b) Assume X is strictly positive. Define $Y_t = \log X_t$. Show that

$$\mathrm{d}Y = \frac{\mathrm{d}X}{X} - \frac{1}{2}\left(\frac{\mathrm{d}X}{X}\right)^2 .$$

Solution: For $f(x) = \log x$, we have $f'(x) = 1/x$ and $f''(x) = -1/x^2$, so

$$\begin{aligned} \mathrm{d}Y &= f'(X)\,\mathrm{d}X + \frac{1}{2}f''(X)(\mathrm{d}X)^2 \\ &= \frac{\mathrm{d}X}{X} - \frac{1}{2}\frac{(\mathrm{d}X)^2}{X^2} . \end{aligned}$$

(c) Assume X is strictly positive. Define $Y_t = X_t^{-\lambda}$ for a constant λ. Show that

$$\frac{\mathrm{d}Y}{Y} = -\lambda\frac{\mathrm{d}X}{X} + \frac{\lambda(1+\lambda)}{2}\left(\frac{\mathrm{d}X}{X}\right)^2 .$$

Solution: For $y = f(x) = x^{-\lambda}$, we have $f'(x) = -\lambda x^{-\lambda-1} = -\lambda y/x$ and $f''(x) = \lambda(1+\lambda))x^{-\lambda-2} = \lambda(1+\lambda)y/x^2$. Therefore,

$$\begin{aligned} \mathrm{d}Y &= f'(X)\,\mathrm{d}X + \frac{1}{2}f''(X)(\mathrm{d}X)^2 \\ &= -\lambda\frac{Y}{X}\,\mathrm{d}X + \frac{1}{2}\lambda(1+\lambda)\frac{Y}{X^2}(\mathrm{d}X)^2 . \end{aligned}$$

12.3. Assume X_1 and X_2 are strictly positive Itô processes. Use Itô's formula to derive the following.

(a) Define $Y_t = X_{1t}X_{2t}$. Show that

$$\frac{\mathrm{d}Y}{Y} = \frac{\mathrm{d}X_1}{X_1} + \frac{\mathrm{d}X_2}{X_2} + \left(\frac{\mathrm{d}X_1}{X_1}\right)\left(\frac{\mathrm{d}X_2}{X_2}\right).$$

Solution: For $f(x_1, x_2) = x_1 x_2$, we have

$$\frac{\partial f}{\partial x_1} = x_2\,, \quad \frac{\partial f}{\partial x_2} = x_1\,, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 1\,, \quad \frac{\partial^2 f}{\partial x_i^2} = 0\,.$$

Hence,

$$\mathrm{d}Y = X_2\,\mathrm{d}X_1 + X_1\,\mathrm{d}X_2 + (\mathrm{d}X_1)(\mathrm{d}X_2)\,.$$

This implies

$$\frac{\mathrm{d}Y}{Y} = \frac{\mathrm{d}X_1}{X_1} + \frac{\mathrm{d}X_2}{X_2} + \left(\frac{\mathrm{d}X_1}{X_1}\right)\left(\frac{\mathrm{d}X_2}{X_2}\right).$$

(b) Define $Y_t = X_{1t}/X_{2t}$. Show that

$$\frac{\mathrm{d}Y}{Y} = \frac{\mathrm{d}X_1}{X_1} - \frac{\mathrm{d}X_2}{X_2} - \left(\frac{\mathrm{d}X_1}{X_1}\right)\left(\frac{\mathrm{d}X_2}{X_2}\right) + \left(\frac{\mathrm{d}X_2}{X_2}\right)^2.$$

Solution: For $f(x_1, x_2) = x_1/x_2$, we have

$$\frac{\partial f}{\partial x_1} = \frac{1}{x_2}\,, \quad \frac{\partial f}{\partial x_2} = -\frac{x_1}{x_2^2}\,, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = -\frac{1}{x_2^2}\,, \quad \frac{\partial^2 f}{\partial x_1^2} = 0\,, \quad \frac{\partial^2 f}{\partial x_2^2} = 2\frac{x_1}{x_2^3}\,.$$

Hence,

$$\mathrm{d}Y = \frac{1}{X_2}\,\mathrm{d}X_1 - \frac{X_1}{X_2^2}\,\mathrm{d}X_2 - \frac{1}{X_2^2}(\mathrm{d}X_1)(\mathrm{d}X_2) + \frac{X_1}{X_2^3}(\mathrm{d}X_2)^2\,.$$

This implies

$$\frac{\mathrm{d}Y}{Y} = \frac{X_2}{X_1}\,\mathrm{d}Y = \frac{\mathrm{d}X_1}{X_1} - \frac{\mathrm{d}X_2}{X_2} - \left(\frac{\mathrm{d}X_1}{X_1}\right)\left(\frac{\mathrm{d}X_2}{X_2}\right) + \left(\frac{\mathrm{d}X_2}{X_2}\right)^2.$$

12.4. Assume S is a geometric Brownian motion:

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

for constants μ and σ and a Brownian motion B.

(a) Show that

$$\mathrm{var}_t\left(\frac{S_{t+1}}{S_t}\right) = \mathrm{e}^{2\mu}\left(\mathrm{e}^{\sigma^2} - 1\right).$$

Hint: Compare Exercise 1.7.

Solution: We have

$$\begin{aligned}\log S_{t+1} &= \log S_t + \int_t^{t+1}\left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}s + \int_t^{t+1}\sigma\,\mathrm{d}B_s \\ &= \log S_t + \mu - \frac{1}{2}\sigma^2 + \sigma(B_u - B_t)\,.\end{aligned}$$

Hence, $\log S_{t+1} - \log S_t$ is normally distributed with mean $\mu - \sigma^2/2$ and variance σ^2. Thus, S_{t+1}/S_t is the exponential of a normally distributed variable with mean $\mu-\sigma^2/2$ and variance σ^2. This is the same assumption as in Exercise 1.7 (with μ in that exercise being $\mu - \sigma^2/2$ here). Thus,

$$\begin{aligned}\mathrm{var}_t\left(\frac{S_{t+1}}{S_t}\right) &= \mathsf{E}_t\left[\frac{S_{t+1}}{S_t}\right]^2\left(\mathrm{e}^{\sigma^2} - 1\right) \\ &= \mathrm{e}^{2\mu}\left(\mathrm{e}^{\sigma^2} - 1\right).\end{aligned}$$

(b) Use the result of the previous part, the formula (12.22), and the approximation $\mathrm{e}^x \approx 1 + x$ to derive approximate formulas for $\mathrm{var}_t(S_{t+1}/S_t)$ and $\mathsf{E}_t[S_{t+1}/S_t]$.

Solution: From (12.22), we have

$$\mathsf{E}\left[\frac{S_{t+1}}{S_t}\right] = \mathrm{e}^{\mu} \approx 1 + \mu\,.$$

We can also say

$$\mathrm{var}\left(\frac{S_{t+1}}{S_t}\right) \approx (1 + 2\mu)\sigma^2\,.$$

12.5. Assume

$$X_t = \theta - \mathrm{e}^{-\kappa t}(\theta - X_0) + \sigma \int_0^t \mathrm{e}^{-\kappa(t-s)}\,\mathrm{d}B_s$$

for a Brownian motion B and constants θ and κ. Show that

$$\mathrm{d}X = \kappa(\theta - X)\,\mathrm{d}t + \sigma\,\mathrm{d}B\,.$$

Note: The process X is called an Ornstein-Uhlenbeck process. Assuming $\kappa > 0$, θ is called the long-run or unconditional mean, and κ is the rate of mean reversion. This is the interest rate process in the Vasicek model (Section 18.3).

Solution: Define

$$Z_t = \sigma \int_0^t \mathrm{e}^{\kappa s}\,\mathrm{d}B_s$$

and

$$f(t,z) = \theta - \mathrm{e}^{-\kappa t}(\theta - X_0) + \mathrm{e}^{-\kappa t} z\,.$$

Then, $X_t = f(t, Z_t)$ and $\mathrm{d}Z_t = \sigma \mathrm{e}^{\kappa t}\,\mathrm{d}B_t$. We have

$$\frac{\partial f}{\partial t} = \kappa \mathrm{e}^{-\kappa t}(\theta - X_0 - z)\,, \quad \frac{\partial f}{\partial z} = \mathrm{e}^{-\kappa t}\,, \quad \frac{\partial^2 f}{\partial z^2} = 0\,.$$

Hence,

$$\begin{aligned} \mathrm{d}X_t &= \kappa \mathrm{e}^{-\kappa t}(\theta - X_0 - Z_t)\,\mathrm{d}t + \mathrm{e}^{-\kappa t}\,\mathrm{d}Z_t \\ &= \kappa(\theta - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t\,. \end{aligned}$$

12.6. Let X be an Ornstein-Uhlenbeck process with a long-run mean of zero; that is,

$$\mathrm{d}X = -\kappa X\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

for constants κ and σ. Set $Y = X^2$. Show that

$$\mathrm{d}Y = \widehat{\kappa}(\widehat{\theta} - Y)\,\mathrm{d}t + \widehat{\sigma}\sqrt{Y}\,\mathrm{d}B$$

for constants $\widehat{\kappa}$, $\widehat{\theta}$ and $\widehat{\sigma}$. Note: The squared Ornstein-Uhlenbeck process Y is a special case of the interest rate process in the Cox-Ingersoll-Ross model (Section 18.3) and a special case of the variance process in the Heston model (Section 17.4)—special because $\widehat{\kappa}\widehat{\theta} = \widehat{\sigma}^2/4$.

Solution: For $f(x) = x^2$, we have $f'(x) = 2x$ and $f''(x) = 2$, so

$$\begin{aligned}
\mathrm{d}Y &= 2X\,\mathrm{d}X + (\mathrm{d}X)^2 \\
&= -2\kappa X^2\,\mathrm{d}t + 2\sigma X\,\mathrm{d}B + \sigma^2\,\mathrm{d}t \\
&= (\sigma^2 - 2\kappa Y)\,\mathrm{d}t + 2\sigma\sqrt{Y}\,\mathrm{d}B \\
&= 2\kappa\left(\frac{\sigma^2}{2\kappa} - Y\right)\,\mathrm{d}t + 2\sigma\sqrt{Y}\,\mathrm{d}B \\
&= \hat{\kappa}(\hat{\theta} - Y)\,\mathrm{d}t + \hat{\sigma}\sqrt{Y}\,\mathrm{d}B\,,
\end{aligned}$$

where we define

$$\hat{\kappa} = 2\kappa\,, \quad \hat{\theta} = \frac{\sigma^2}{2\kappa}\,, \quad \hat{\sigma} = 2\sigma\,.$$

12.7. Suppose $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$ for constants μ and σ and a Brownian motion B. Let r be a constant. Consider a wealth process W as defined in Section 12.2:

$$\frac{\mathrm{d}W}{W} = (1-\pi)r\,\mathrm{d}t + \pi\frac{\mathrm{d}S}{S}\,,$$

where π is a constant.

(a) By observing that W is a geometric Brownian motion, derive an explicit formula for W_t.

Solution: We have

$$\frac{\mathrm{d}W}{W} = [r + \pi(\mu - r)]\,\mathrm{d}t + \pi\sigma\,\mathrm{d}B\,.$$

Thus,

$$W_t = W_0\exp\left(\left(r + \pi(\mu - r) - \frac{1}{2}\pi^2\sigma^2\right)t + \pi\sigma B_t\right)\,.$$

(b) For a constant ρ and dates $s < t$, calculate $\mathsf{E}_s[W_t^{1-\rho}]$. Hint: write $W_t^{1-\rho} = \mathrm{e}^{(1-\rho)\log W_t}$.

Solution: For any $s < t$, we have

$$\log W_t = \log W_s + \left(r + \pi(\mu - r) - \frac{1}{2}\pi^2\sigma^2\right)(t - s) + \pi\sigma(B_t - B_s)\,.$$

Hence,

$$\begin{aligned} W_t^{1-\rho} &= \mathrm{e}^{(1-\rho)\log W_t} \\ &= W_s^{1-\rho}\exp\left((1-\rho)\left(r + \pi(\mu - r) - \frac{1}{2}\pi^2\sigma^2\right)(t - s) + (1-\rho)\pi\sigma(B_t - B_s)\right)\,. \end{aligned}$$

By the usual rule for means of exponentials of normals and the fact that $B_t - B_s$ is normally distributed with mean zero and variance $t - s$, this implies

$$\begin{aligned} \mathsf{E}_s\left[W_t^{1-\rho}\right] &= W_s^{1-\rho}\exp\left((1-\rho)\left(r + \pi(\mu - r) - \frac{1}{2}\pi^2\sigma^2\right)(t - s) + \frac{1}{2}(1-\rho)^2\pi^2\sigma^2(t - s)\right) \\ &= W_s^{1-\rho}\exp\left((1-\rho)\left(r + \pi(\mu - r) - \frac{1}{2}\pi^2\sigma^2 + \frac{1}{2}(1-\rho)\pi^2\sigma^2\right)(t - s)\right) \\ &= W_s^{1-\rho}\exp\left((1-\rho)\left(r + \pi(\mu - r) - \frac{1}{2}\rho\pi^2\sigma^2\right)(t - s)\right)\,. \end{aligned}$$

(c) Consider an investor who chooses a portfolio process to maximize

$$\mathsf{E}\left[\frac{1}{1-\rho}w_T^{1-\rho}\right]\,.$$

Show that if a constant portfolio $\pi_t = \pi$ is optimal, then the optimal portfolio is

$$\pi = \frac{\mu - r}{\rho\sigma^2}\,.$$

Solution: The objective is to maximize

$$\frac{1}{1-\rho}\exp\left(T(1-\rho)\left(r + \pi(\mu - r) - \frac{1}{2}\rho\pi^2\sigma^2\right)\right)\,.$$

The first-order condition is

$$\frac{1}{1-\rho}\exp\left(T(1-\rho)\left(r + \pi(\mu - r) - \frac{1}{2}\rho\pi^2\sigma^2\right)\right)T(1-\rho)\left(\mu - r - \rho\sigma^2\pi\right) = 0\,.$$

Equivalently, $\mu - r - \rho\sigma^2\pi = 0$, which implies $\pi = (\mu - r)/\rho\sigma^2$.

12.8. Let B be a Brownian motion. Define $Y_t = B_t^2 - t$.

(a) Use the fact that a Brownian motion has independent zero-mean increments with variance equal to the length of the time interval to show that Y is a martingale.

Solution: We want to show that $\mathsf{E}_s[B_t^2 - t] = B_s^2 - s$. This is equivalent to $\mathsf{E}_s[B_t^2] - B_s^2 = t - s$. We know that $\text{var}_s(B_t) = t - s$, so it suffices to show that $\text{var}_s(B_t) = \mathsf{E}_s[B_t^2] - B_s^2$. This follows from the definition of variance and the fact that $\mathsf{E}_s[B_t] = B_s$.

(b) Apply Itô's formula to calculate $\mathrm{d}Y$ and verify condition (12.5) to show that Y is a martingale. Hint: To verify (12.5) use the fact that

$$\mathsf{E}\left[\int_0^T B_t^2 \,\mathrm{d}t\right] = \int_0^T \mathsf{E}[B_t^2]\,\mathrm{d}t\,.$$

Solution: For $f(t,x) = x^2 - t$, we have

$$\frac{\partial f}{\partial t} = -1\,, \quad \frac{\partial f}{\partial x} = 2x\,, \quad \frac{\partial^2 f}{\partial x^2} = 2\,.$$

Thus,

$$\mathrm{d}Y_t = \mathrm{d}f(t, B_t) = -\,\mathrm{d}t + 2B_t\,\mathrm{d}B_t + (\mathrm{d}B)^2 = 2B_t\,\mathrm{d}B_t\,.$$

To verify that Y is a martingale on $[0,T]$, we need to show that

$$\mathsf{E}\int_0^T (2B_t)^2\,\mathrm{d}t < \infty\,.$$

We have

$$\mathsf{E}\int_0^T (2B_t)^2\,\mathrm{d}t = 4\int_0^T \mathsf{E}[B_t^2]\,\mathrm{d}t = 4\int_0^T t\,\mathrm{d}t = 2T^2 < \infty\,.$$

(c) Let $\mathrm{d}M = \theta\,\mathrm{d}B$ for a Brownian motion B. Use Itô's formula to show that

$$M_t^2 - \int_0^t (\mathrm{d}M_s)^2$$

is a local martingale.

Solution: Set $Z_t = M_t^2$ and

$$Y_t = Z_t - \int_0^t (\mathrm{d}M_s)^2 .$$

Then,

$$\mathrm{d}Y_t = \mathrm{d}Z_t - (\mathrm{d}M_t)^2 .$$

As in the preceding part, applying Itô's formula to $Z_t = f(M_t) = M_t^2$ gives

$$\mathrm{d}Z_t = 2M_t\,\mathrm{d}M_t + (\mathrm{d}M_t)^2 .$$

Thus, $\mathrm{d}Y_t = 2M_t\,\mathrm{d}M_t$, which inherits the local martingale property of M.

(d) Let $\mathrm{d}M_i = \theta_i\,\mathrm{d}B_i$ for $i = 1, 2$, and Brownian motions B_1 and B_2. Use Itô's formula to show that

$$M_{1t}M_{2t} - \int_0^t (\mathrm{d}M_{1s})\,(\mathrm{d}M_{2s})$$

is a local martingale.

Solution: Set $Z_t = M_{1t}M_{2t}$ and

$$Y_t = Z_t - \int_0^t (\mathrm{d}M_{1s})\,(\mathrm{d}M_{2s}) .$$

Then

$$\mathrm{d}Y_t = \mathrm{d}Z_t - (\mathrm{d}M_{1t})\,(\mathrm{d}M_{2t}) .$$

Applying Itô's formula to Z_t as in Exercise 12.3 gives

$$\mathrm{d}Z_t = M_{1t}\,\mathrm{d}M_{2t} + M_{2t}\,\mathrm{d}M_{1t} + (\mathrm{d}M_{1t})\,(\mathrm{d}M_{2t}) .$$

Thus,

$$\mathrm{d}Y_t = M_{1t}\,\mathrm{d}M_{2t} + M_{2t}\,\mathrm{d}M_{1t} ,$$

implying Y is a local martingale.

12.9. Let B_1 and B_2 be independent Brownian motions and let $\rho \in [-1, 1]$. Set $\widehat{B}_1 = B_1$. Define $\widehat{B}_2$ by $\widehat{B}_{20} = 0$ and $\mathrm{d}\widehat{B}_2 = \rho\,\mathrm{d}B_1 + \sqrt{1-\rho^2}\,\mathrm{d}B_2$.

(a) Use Levy's theorem to show that $\widehat{B}_2$ is a Brownian motion.

Solution: $\widehat{B}_2$ is a continuous local martingale with

$$(\mathrm{d}\widehat{B}_2)^2 = \rho^2(\mathrm{d}B_1)^2 + (1-\rho^2)(\mathrm{d}B_2)^2 + 2\rho\sqrt{1-\rho^2}(\mathrm{d}B_1)(\mathrm{d}B_2) = \rho^2\,\mathrm{d}t + (1-\rho^2)\,\mathrm{d}t = \mathrm{d}t\,.$$

Thus, $\widehat{B}_2$ is a Brownian motion.

(b) Show that ρ is the correlation process of the two Brownian motions $\widehat{B}_1$ and $\widehat{B}_2$.

Solution: We have

$$(\mathrm{d}\widehat{B}_1)(\mathrm{d}\widehat{B}_2) = (\mathrm{d}B_1)(\rho\,\mathrm{d}B_1 + \sqrt{1-\rho^2}\,\mathrm{d}B_2) = \rho\,\mathrm{d}t\,.$$

12.10. Let $\rho \neq \pm\,1$ be the correlation process of two Brownian motions B_1 and B_2. Set $\widehat{B}_1 = B_1$. Define $\widehat{B}_2$ by $\widehat{B}_{20} = 0$ and

$$\mathrm{d}\widehat{B}_2 = \frac{1}{\sqrt{1-\rho^2}}(\mathrm{d}B_2 - \rho\,\mathrm{d}B_1)\,.$$

Show that $\widehat{B}_1$ and $\widehat{B}_2$ are independent Brownian motions. Note: Obviously this reverses the process of the previous exercise. It gives us

$$\mathrm{d}B_2 = \rho\,\mathrm{d}\widehat{B}_1 + \sqrt{1-\rho^2}\,\mathrm{d}\widehat{B}_2\,,$$

so $\rho\,\mathrm{d}\widehat{B}_1$ can be viewed as the orthogonal projection of $\mathrm{d}B_2$ on $\mathrm{d}B_1 = \mathrm{d}\widehat{B}_1$.

Solution: The following shows that $\widehat{B}_2$ is a Brownian motion:

$$\begin{aligned}
(\mathrm{d}\widehat{B}_2)^2 &= \frac{1}{1-\rho^2}\left[(\mathrm{d}B_2)^2 - 2\rho(\mathrm{d}B_2)(\mathrm{d}B_1) + \rho^2(\mathrm{d}B_1)^2\right] \\
&= \frac{1}{1-\rho^2}\left[\mathrm{d}t - 2\rho^2\,\mathrm{d}t + \rho^2\,\mathrm{d}t\right] \\
&= \mathrm{d}t\,.
\end{aligned}$$

The following shows that $\widehat{B}_1$ and $\widehat{B}_2$ are independent:

$$\begin{aligned}(\mathrm{d}\widehat{B}_1)(\mathrm{d}\widehat{B}_2) &= \frac{1}{\sqrt{1-\rho^2}}(\mathrm{d}B_1)(\mathrm{d}B_2 - \rho\,\mathrm{d}B_1)\\ &= \frac{1}{\sqrt{1-\rho^2}}(\rho\,\mathrm{d}t - \rho\,\mathrm{d}t)\\ &= 0\,.\end{aligned}$$

12.11. Let B_1 and B_2 be independent Brownian motions and

$$\mathrm{d}Z \stackrel{\text{def}}{=} \begin{pmatrix}\mathrm{d}Z_1\\ \mathrm{d}Z_2\end{pmatrix} = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{pmatrix}\begin{pmatrix}\mathrm{d}B_1\\ \mathrm{d}B_2\end{pmatrix} \stackrel{\text{def}}{=} A\,\mathrm{d}B$$

for stochastic processes σ_{ij}, where A is the matrix of the σ_{ij}.

(a) Calculate a, b and c with $a > 0$ and $c > 0$ such that $LL' = AA'$, where

$$L = \begin{pmatrix}a & 0\\ b & c\end{pmatrix}.$$

Solution: We have

$$(\mathrm{d}Y_1)^2 = (\sigma_{11}^2 + \sigma_{12}^2)\,\mathrm{d}t\,, \quad (\mathrm{d}Y_1)(\mathrm{d}Y_2) = (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})\,\mathrm{d}t\,, \quad (\mathrm{d}Y_2)^2 = (\sigma_{21}^2 + \sigma_{22}^2)\,\mathrm{d}t\,.$$

Hence,

$$A = \begin{pmatrix}\sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}\\ \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} & \sigma_{21}^2 + \sigma_{22}^2\end{pmatrix}.$$

(b) We want to solve

$$\begin{pmatrix}a & 0\\ b & c\end{pmatrix}\begin{pmatrix}a & b\\ 0 & c\end{pmatrix} = \begin{pmatrix}\sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}\\ \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} & \sigma_{21}^2 + \sigma_{22}^2\end{pmatrix}.$$