Solution: When returns are normally distributed, a CARA investor chooses the return $\tilde{R}$ that maximizes

$$\mathsf{E}[\tilde{R}] - \frac{1}{2}\alpha w_0 \operatorname{var}(\tilde{R})\,.$$

Given $\tilde{R} = \tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon}$ and Facts 11 and 15, the objective function is

$$\mathsf{E}[\tilde{R} + b\tilde{e}_p] - \frac{1}{2}\alpha w_0[\operatorname{var}(\tilde{R}_p + b\tilde{e}_p) + \operatorname{var}(\tilde{\varepsilon})]\,,$$

so it is optimal to choose $\tilde{\varepsilon} = 0$. The investor chooses b to maximize

$$b\mathsf{E}[\tilde{e}_p] - \frac{1}{2}\alpha w_0[2b\operatorname{cov}(\tilde{R}_p, \tilde{e}_p) + b^2\operatorname{var}(\tilde{e}_p)]\,,$$

and the optimum satisfies

$$\mathsf{E}[\tilde{e}_p] - \alpha w_0 \operatorname{cov}(\tilde{R}_p, \tilde{e}_p) - \alpha w_0 \operatorname{var}(\tilde{e}_p) b = 0\,,$$

implying

$$b = \frac{\mathsf{E}[\tilde{e}_p]}{\alpha w_0 \operatorname{var}(\tilde{e}_p)} - \frac{\operatorname{cov}(\tilde{R}_p, \tilde{e}_p)}{\operatorname{var}(\tilde{e}_p)}\,.$$

Using Facts 8 and 17, we can simplify this further to

$$b = \frac{1 + \alpha w_0 \mathsf{E}[\tilde{R}_p]}{\alpha w_0(1 - \mathsf{E}[\tilde{e}_p])}\,.$$

5.8. Assume there is a risk-free asset.

(a) Using the formula (3.45) for $\tilde{m}_p$, compute λ such that

$$\tilde{R}_p = \lambda \pi_{\text{tang}}' \tilde{\mathbf{R}} + (1 - \lambda) R_f.$$

Solution: We have

$$\tilde{m}_p = \frac{1}{R_f} + \left(\iota - \frac{1}{R_f}\mu\right)' \Sigma^{-1}(\tilde{\mathbf{R}} - \mu)\,.$$

Hence

$$\text{var}(\tilde{m}_p) = \frac{\kappa^2}{R_f^2}\,,$$

where $\kappa^2 = (R_f\iota - \mu)'\Sigma^{-1}(R_f\iota - \mu)$ is the squared maximum Sharpe ratio. Because $\mathsf{E}[\tilde{m}_p] = 1/R_f$, this implies

$$\mathsf{E}[\tilde{m}_p^2] = \frac{1+\kappa^2}{R_f^2}\,.$$

Therefore, by the definition $\tilde{R}_p = \tilde{m}_p/\mathsf{E}[\tilde{m}_p^2]$, we have

$$\tilde{R}_p = \frac{R_f}{1+\kappa^2} + \frac{R_f}{1+\kappa^2}\,(R_f\iota - \mu)'\,\Sigma^{-1}(\tilde{\mathbf{R}} - \mu)\,,$$

in the notation of Section 5.2. Setting

$$\lambda = -\frac{R_f(B - R_f C)}{1+\kappa^2}\,,$$

we have

$$1 - \lambda = \frac{1+\kappa^2 + R_f B - R_f^2 C}{1+\kappa^2} = \frac{1 + A - R_f B}{1+\kappa^2}\,,$$

because $\kappa^2 = A - 2R_f B + R_f^2 C$. Thus,

$$\tilde{R}_p = \lambda \pi_{\text{tang}}' \tilde{\mathbf{R}} + (1-\lambda) R_f\,.$$

(b) Show that λ in part (a) is negative when $R_f < B/C$ and positive when $R_f > B/C$. Note: This shows that $\tilde{R}_p$ is on the inefficient part of the frontier, because the portfolio generating $\tilde{R}_p$ is short the tangency portfolio when the tangency portfolio is efficient and long the tangency portfolio when it is inefficient.

Solution:

$$\lambda = -\frac{R_f(B - R_f C)}{1+\kappa^2} < 0$$

when $B > R_f C$ and positive when $B < R_f C$.

5.9. Consider the problem of choosing a portfolio π of risky assets, a proportion $\phi_b \geq 0$ to borrow and a proportion $\phi_\ell \geq 0$ to lend to maximize the expected return $\pi'\mu + \phi_\ell R_\ell - \phi_b R_b$ subject to the constraints $(1/2)\pi'\Sigma\pi \leq k$ and $\iota'\pi + \phi_\ell - \phi_b = 1$. Assume $B/C > R_b > R_\ell$, where B and C are defined in (**??**). Define

$$\pi_b = \frac{1}{\iota'\Sigma^{-1}(\mu - R_b\iota)}\Sigma^{-1}(\mu - R_b\iota)\,,$$
$$\pi_\ell = \frac{1}{\iota'\Sigma^{-1}(\mu - R_\ell\iota)}\Sigma^{-1}(\mu - R_\ell\iota)\,.$$

Using the Kuhn-Tucker conditions, show that the solution is either (i) $\pi = (1-\phi_\ell)\pi_\ell$ for $0 \leq \phi_\ell \leq 1$, (ii) $\pi = \lambda\pi_\ell + (1-\lambda)\pi_b$ for $0 \leq \lambda \leq 1$, or (iii) $\pi = (1+\phi_b)\pi_b$ for $\phi_b \geq 0$.

Solution: The Kuhn-Tucker conditions are

$$\begin{aligned}
\mu - \delta\Sigma\pi - \gamma\iota &= 0\,, \\
R_\ell - \gamma + \eta_\ell &= 0\,, \\
-R_b + \gamma + \eta_b &= 0\,, \\
\phi_\ell, \phi_b, \eta_\ell, \eta_b, \delta &\geq 0\,, \\
\frac{1}{2}\pi'\Sigma\pi &\leq k\,, \\
\iota'\pi + \phi_\ell - \phi_b &= 1\,, \\
\eta_\ell\phi_\ell = \eta_b\phi_b = \delta\left(\frac{1}{2}\pi'\Sigma\pi - k\right) &= 0\,.
\end{aligned}$$

There are three possibilities to consider: (i) $\phi_\ell > 0$, (ii), $\phi_b > 0$, (iii) $\phi_\ell = \phi_b = 0$.

(i) If $\phi_\ell > 0$, then $\eta_\ell = 0$, $\gamma = R_\ell$, and

$$\pi = \frac{1}{\delta}\Sigma^{-1}(\mu - R_\ell\iota)\,.$$

Also, $\gamma = R_\ell$ implies $\eta_b = R_b - R_\ell > 0$. Hence, $\phi_b = 0$, and $\iota'\pi = 1 - \phi_\ell$. This implies $\pi = (1-\phi_\ell)\pi_\ell$.

(ii) If $\phi_b > 0$, then $\eta_b = 0$, $\gamma = -R_b$, and

$$\pi = \frac{1}{\delta}\Sigma^{-1}(\mu - R_b\iota)\,.$$

Also, $\gamma = -R_b$ implies $\eta_\ell = R_b - R_\ell > 0$, so $\phi_\ell = 0$. This implies $\iota'\pi = 1+\phi_b$. Hence, $\pi = (1+\phi_b)\pi_b$.

(iii) If $\phi_\ell = \phi_b = 0$, then

$$\pi = \frac{1}{\delta}\Sigma^{-1}(\mu - \gamma\iota)\,,$$

where $\gamma = R_\ell + \eta_\ell \geq R_\ell$ and $\gamma = R_b - \eta_b \leq R_b$. Thus, $\gamma = \lambda R_\ell + (1-\lambda)R_b$ for some $0 \leq \lambda \leq 1$. From $\iota'\pi = 1$, it follows that $\pi = \lambda\pi_\ell + (1-\lambda)\pi_b$.

Chapter 6

Factor Models

6.1. Assume there exists a return $\tilde{R}_*$ that is on the mean-variance frontier and is an affine function of a vector $\tilde{F}$; that is, $\tilde{R} = a + b'\tilde{F}$. Assume either (i) there is a risk-free asset and $\tilde{R}_* \neq R_f$, or (ii) there is no risk-free asset and $\tilde{R}_*$ is different from the GMV return. Show that there is a factor model with factors $\tilde{F}$.

Solution: Under either assumption (i) or (ii), there is a factor model with $\tilde{R}^*$ as the factor. Thus, for every return $\tilde{R}$,

$$\begin{aligned}\mathsf{E}[\tilde{R}] &= \alpha + \lambda \operatorname{cov}(\tilde{R}^*, \tilde{R}) \\ &= \alpha + \lambda b' \operatorname{Cov}(\tilde{F}, \tilde{R}) \\ &= \alpha + \hat{\lambda}' \Sigma_F^{-1} \operatorname{Cov}(\tilde{F}, \tilde{R}),\end{aligned}$$

where we define

$$\hat{\lambda} = \lambda \Sigma_F b\,.$$

6.2. Assume returns are normally distributed, investors have CARA utility,and there is no labor income. Derive the CAPM from the portfolio formula (2.22), that is, from

$$\phi_h = \frac{1}{\alpha_h}\Sigma^{-1}(\mu - R_f\iota)\,,$$

where α_h denotes the absolute risk aversion of investor h. Show that the price of risk is $\alpha w_0 \operatorname{var}(\tilde{R}_m)$, where α is the aggregate absolute risk aversion defined in Section 1.1 and $w_0 = \iota' \sum_{h=1}^H \phi_h$ is the market value of risky assets at date 0.

Solution: Setting $\phi = \sum_{h=1}^H \phi_h$ and adding the optimal portfolios over investors gives

$$\phi = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f\iota)\,,$$

which in equilibrium is the vector of market values of risky assets at date 0. Thus,

$$\mu = R_f\iota + \alpha\Sigma\phi\,.$$

The market portfolio is $\pi = (1/\iota'\phi)\phi$. The return on the market portfolio is $\tilde{R}_m = \phi'\tilde{R}^{\text{vec}}/\phi'\iota$, and the vector of betas of the asset returns with respect to the market return is

$$\frac{1}{\pi'\Sigma\pi}\Sigma\pi\,.$$

We have

$$\mu = R_f\iota + \alpha(\iota'\phi)\Sigma\pi = \alpha(\iota'\phi)(\pi'\Sigma\pi)\frac{1}{\pi'\Sigma\pi}\Sigma\pi\,,$$

so the factor risk premium is

$$\alpha(\iota'\phi)(\pi'\Sigma\pi) = \alpha w_0 \operatorname{var}(\tilde{R}_m)\,.$$

6.3. Assume there is a risk-free asset, and assume that a factor model holds in which each factor $\tilde{f}_1, \ldots, \tilde{f}_k$ is an excess return.

(a) Show that each return $\tilde{R}$ on the mean-variance frontier equals

$$R_f + \sum_{j=1}^k \beta_j \tilde{f}_j \tag{6.33}$$

for some $\beta_1, \ldots, \beta_k$. In other words, show that the risk-free return and the factors span the mean-variance frontier.

Solution:

Consider an arbitrary return $\tilde{R}$. Let $a + \sum_{j=1}^k \beta_j \tilde{f}_j$ be the orthogonal projection of $\tilde{R}$ onto the span of the $\tilde{f}_j$ and a constant. This means that

$$\tilde{R} = a + \sum_{j=1}^{k} \beta_j \tilde{f}_j + \tilde{\varepsilon},$$

where $\tilde{\varepsilon}$ has a zero mean and is orthogonal to the $\tilde{f}_j$. Set $\tilde{R}^* = R_f + \sum_{j=1}^k \beta_j \tilde{f}_j$ for the same β_j's. Because the $\tilde{f}_j$ are excess returns, $\tilde{R}^*$ is a return. We have

$$\mathsf{E}[\tilde{R}^*] = R_f + \sum_{j=1}^{k} \beta_j \mathsf{E}[\tilde{f}_j] \,.$$

Because there is factor pricing, we also have

$$\mathsf{E}[\tilde{R}] = R_f + \sum_{j=1}^{k} \beta_j \mathsf{E}[\tilde{f}_j] \,.$$

Therefore, $\mathsf{E}[\tilde{R}^*] = \mathsf{E}[\tilde{R}]$. Furthermore,

$$\operatorname{var}(\tilde{R}) = \operatorname{var}\left(\sum_{j=1}^{k} \beta_j \tilde{f}_j\right) + \operatorname{var}(\tilde{\varepsilon}_i) \geq \operatorname{var}\left(\sum_{j=1}^{k} \beta_j \tilde{f}_j\right) = \operatorname{var}(\tilde{R}^*)\,,$$

with the weak inequality being an equality only if $\tilde{\varepsilon} = 0$ (in which case we must have $a = R_f$). Therefore, for each return $\tilde{R}$ that is not of the form $R_f + \sum_{j=1}^k \beta_j \tilde{f}_j$, there is a return that is of the form $R_f + \sum_{j=1}^k \beta_j \tilde{f}_j$ that has the same mean and a strictly smaller variance. This implies that all frontier returns are of the form $R_f + \sum_{j=1}^k \beta_j \tilde{f}_j$.

(b) Show that a return of the form (6.33) is on the mean-variance frontier if and only if $\beta = (\beta_1 \ \cdots \ \beta_k)'$ satisfies

$$\beta = \delta \Sigma_F^{-1} \lambda$$

for some δ, where Σ_F is the (assumed to be nonsingular) covariance matrix of $\tilde{F} = (\tilde{f}_1 \; \cdots \; \tilde{f}_k)'$, and $\lambda = \mathsf{E}[\tilde{F}] \in \mathbb{R}^k$.

Solution: The frontier consists of the returns of the form (6.33) that have minimum variance for a given mean. That is, the frontier consists of the returns $R_f + \beta'\tilde{F}$ where β solves

$$\min_{\beta} \quad \frac{1}{2}\beta'\Sigma_F\beta \quad \text{subject to} \quad R_f + \beta'\lambda = \mu_{\text{targ}}\,.$$

The first-order condition for this optimization problem is $\Sigma_F\beta = \delta\lambda$ for some Lagrange multiplier δ. Thus, $\beta = \delta\Sigma_F^{-1}\lambda$.

6.4. Suppose there is a risk-free asset and suppose Jensen's alpha in (6.22) is positive. Consider an investor with initial wealth w_0 who holds the benchmark portfolio and therefore has terminal wealth $w_0\tilde{R}_b$. Assume $\mathsf{E}[u'(w_0\tilde{R}_b)] > 0$. Consider the return

$$\tilde{R}_1 = \tilde{R} + (1-\beta)(\tilde{R}_b - R_f) = \tilde{R}_b + \alpha + \tilde{\varepsilon}.$$

Show that

$$\mathsf{E}[u'(w_0\tilde{R}_b)(\tilde{R}_1 - \tilde{R}_b)] = \alpha\mathsf{E}[u'(w_0\tilde{R}_b)] > 0 \tag{6.34}$$

if utility is quadratic or if $\tilde{R}$ and $\tilde{R}_b$ are joint normally distributed. Note: Condition (6.34) implies that the expected utility of a convex combination $\lambda\tilde{R}_1 + (1-\lambda)\tilde{R}_b$ is greater than the expected utility of $\tilde{R}_b$ for sufficiently small $\lambda > 0$. Thus, this exercise shows that a positive Jensen's alpha implies that utility improvements are possible if utility is quadratic or returns are joint normal.

Solution: $\mathsf{E}[u'(w_0\tilde{R}_b)(\tilde{R}_1 - \tilde{R}_b)] = \mathsf{E}[u'(w_0\tilde{R}_b)(\alpha + \tilde{\varepsilon})] = \alpha\mathsf{E}[u'(w_0\tilde{R}_b)] + \mathsf{E}[u'(w_0\tilde{R}_b)\tilde{\varepsilon}]$

By construction, $\tilde{\varepsilon}$ is uncorrelated with $\tilde{R}_b$. In general, we cannot argue that $\mathsf{E}[u'(w_0\tilde{R}_b)\tilde{\varepsilon}] = 0$ but consider the two cases below:

- If u is quadratic then u' is linear and $\mathsf{E}[u'(w_0\tilde{R}_b)\tilde{\varepsilon}] = 0$ follows from $\text{cov}(\tilde{R}_b, \tilde{\varepsilon}) = 0$.

- If $\tilde{R}$ and $\tilde{R}_b$ are joint normally distributed then $\tilde{\varepsilon}$ and $\tilde{R}_b$ are also joint normally distributed and since they are uncorrelated they are independent.

6.5. Suppose investors can borrow and lend at different rates. Let R_b denote the return on borrowing and R_ℓ the return on lending. Suppose $B/C > R_b > R_\ell$, where B and C are defined in (5.6). Suppose each investor chooses a mean-variance efficient portfolio, as described in Exercise 5.9. Show that the CAPM holds with $R_\ell \leq R_z \leq R_b$.

Solution: From Exercise 5.9, the optimal portfolio of risky assets for each investor h is

$$\delta_h[\lambda_h \pi_\ell + (1-\lambda_h)\pi_b]\,,$$

where $0 \leq \lambda_h \leq 1$ and $\delta_h \geq 0$ is the fraction of investor h's wealth that is invested in risky assets. It follows that the market portfolio of risky assets is $\lambda\pi_\ell + (1-\lambda)\pi_b$ for some $0 \leq \lambda \leq 1$. The vector of covariances of the risky asset returns with the market return is therefore

$$\text{Cov}(\tilde{R}^{\text{vec}}, \tilde{R}_m) = \Sigma[\lambda\pi_\ell + (1-\lambda)\pi_b] = \frac{\lambda}{B - CR_b}(\mu - R_b\iota) + \frac{1-\lambda}{B - CR_\ell}(\mu - R_\ell\iota)\,.$$

Define $\theta_b = \lambda/(B - CR_b)$ and $\theta_\ell = (1-\lambda)/(B - CR_\ell)$. Then, we have

$$\mu = \frac{1}{\theta_b + \theta_\ell}\,\text{Cov}(\tilde{R}^{\text{vec}}, \tilde{R}_m) + \frac{\theta_b R_b + \theta_\ell R_\ell}{\theta_b + \theta_\ell}\iota\,.$$

Thus,

$$R_z = \frac{\theta_b R_b + \theta_\ell R_\ell}{\theta_b + \theta_\ell}\,,$$

which is a convex combination of R_b and R_ℓ.

6.6. Assume the asset returns $\tilde{R}_i$ for $i = 1, \ldots, n$ satisfy

$$\tilde{R}_i = \mathsf{E}[\tilde{R}_i] + \text{Cov}(\tilde{F}, \tilde{R}_i)'\Sigma_F^{-1}(\tilde{F} - \mathsf{E}[\tilde{F}]) + \tilde{\varepsilon}_i\,,$$

where each $\tilde{\varepsilon}_i$ is mean-independent of the factors $\tilde{F}$, that is, $\mathsf{E}[\tilde{\varepsilon}_i \mid \tilde{F}] = 0$ (note it is not being assumed that $\text{cov}(\tilde{\varepsilon}_i, \tilde{\varepsilon}_j) = 0$). Assume markets are complete and the market return is well diversified in the sense of having no idiosyncratic risk:

$$\tilde{R}_m = \mathsf{E}[\tilde{R}_m] + \text{Cov}(\tilde{F}, \tilde{R}_m)'\Sigma_F^{-1}(\tilde{F} - \mathsf{E}[\tilde{F}])\,.$$

Show that there is a factor model with factors $\tilde{F}$. Hint: Pareto optimality implies sharing rules $\tilde{w}_h = f_h(\tilde{w}_m)$.

Solution: Complete markets and strictly monotone preferences imply Pareto optimality. Given the sharing rules, the marginal utility of any investor depends only on $\tilde{w}_m$ and hence depends only on $\tilde{F}$. Thus, there is an SDF that is proportional to an investor's marginal utility and equal to $\tilde{m} = g(\tilde{F})$ for some function g. By the strict monotonicity of preferences, $\tilde{m}$ is strictly positive. Hence, $\mathsf{E}[\tilde{m}] \neq 0$. This implies a factor pricing model with $\tilde{m}$ as the factor. This further implies an APT pricing formula with pricing errors $-\mathsf{E}[\tilde{m}\tilde{\varepsilon}_i]/\mathsf{E}[\tilde{m}]$. The pricing errors are zero, because,

$$\mathsf{E}[\tilde{m}\tilde{\varepsilon}_i] = \mathsf{E}[\mathsf{E}[g(\tilde{F})\tilde{\varepsilon}_i \mid \tilde{F}] = \mathsf{E}[g(\tilde{F})\mathsf{E}[\tilde{\varepsilon}_i \mid \tilde{F}] = 0\,.$$

Thus, there is exact APT pricing, i.e., a factor pricing model with $\tilde{F}$ as the factors.

6.7. Suppose two assets satisfy a statistical factor model with a single factor:

$$\tilde{R}_1 = \mathsf{E}[\tilde{R}_1] + \tilde{f} + \tilde{\varepsilon}_1\,,$$

$$\tilde{R}_2 = \mathsf{E}[\tilde{R}_2] - \tilde{f} + \tilde{\varepsilon}_2$$

where $\mathsf{E}[\tilde{f}] = \mathsf{E}[\tilde{\varepsilon_1}] = \mathsf{E}[\tilde{\varepsilon}_2] = 0$, $\text{var}(\tilde{f}) = 1$, $\text{cov}(\tilde{f}, \tilde{\varepsilon}_1) = \text{cov}(\tilde{f}, \tilde{\varepsilon}_2) = 0$, and $\text{cov}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) = 0$. Assume $\text{var}(\tilde{\varepsilon}_1) = \text{var}(\tilde{\varepsilon}_2) = \sigma^2$. Define $\tilde{R}_1^* = \tilde{R}_1$ and $\tilde{R}_2^* = \pi\tilde{R}_1 + (1-\pi)\tilde{R}_2$ with $\pi = 1/(2+\sigma^2)$.

(a) Show that $\tilde{R}_1^*$ and $\tilde{R}_2^*$ do not satisfy a statistical factor model with the single factor $\tilde{f}$.

Solution: $\tilde{R}_2^* = \mathsf{E}[\tilde{R}_2^*] + (2\pi - 1)\tilde{f} + \tilde{\varepsilon}_2^*$, where $\tilde{\varepsilon}_2^* = \pi\tilde{\varepsilon}_1 + (1-\pi)\tilde{\varepsilon}_2$ is uncorrelated with $\tilde{f}$. However, $\text{cov}(\tilde{\varepsilon}_1, \tilde{\varepsilon}_2^*) = \pi\sigma^2 \neq 0$.

(b) Show that $\tilde{R}_1^*$ and $\tilde{R}_2^*$ satisfy a statistical factor model with zero factors, that is,

$$\tilde{R}_1^* = \mathsf{E}[\tilde{R}_1^*] + \tilde{\varepsilon}_1^*\,,$$

$$\tilde{R}_2^* = \mathsf{E}[\tilde{R}_2^*] + \tilde{\varepsilon}_2^*\,,$$

where $\mathsf{E}[\tilde{\varepsilon}_1^*] = \mathsf{E}[\tilde{\varepsilon}_2^*] = 0$ and $\text{cov}(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*) = 0$.

Solution: Define $\tilde{\varepsilon}_1^* = f + \tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2^* = (2\pi - 1)f + \pi\tilde{\varepsilon}_1 + (1-\pi)\tilde{\varepsilon}_2$. Then

$$\tilde{R}_1^* = \mathsf{E}[\tilde{R}_1^*] + \tilde{\varepsilon}_1^*\,,$$

$$\tilde{R}_2^* = \mathsf{E}[\tilde{R}_2^*] + \tilde{\varepsilon}_2^*\,,$$

and $\text{cov}(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*) = (2\pi - 1)\,\text{var}(\tilde{f}) + \pi\sigma^2 = 2\pi - 1 + \pi\sigma^2 = 0$.

(c) Assume exact APT pricing with nonzero risk premium λ for the two assets in the single factor model, that is, $\mathsf{E}[\tilde{R}_i] - R_f = \lambda\,\text{cov}(\tilde{R}_i, \tilde{f})$ for $i = 1, 2$. Show that there cannot be exact APT pricing in the zero factor model for $\tilde{R}_1^*$ and $\tilde{R}_2^*$.

Solution: Exact APT pricing with zero factors means $\mathsf{E}[\tilde{R}_1^*] = \mathsf{E}[\tilde{R}_2^*] = R_f$, but $\mathsf{E}[\tilde{R}_1^*] = R_f + \lambda$.

6.8. Assume there are H investors with CARA utility and the same absolute risk aversion α. Assume there is a risk-free asset. Assume there are two risky assets with payoffs $\tilde{x}_i$ that are joint normally distributed with mean vector μ and nonsingular covariance matrix Σ. Assume H_U investors are unaware of the second asset and invest only in the risk-free asset and the first risky asset. If all investors invested in both assets ($H_U = 0$), then the equilibrium price vector would be

$$p^* = \frac{1}{R_f}\mu - \frac{\alpha}{HR_f}\Sigma\bar{\theta}\,,$$

where $\bar{\theta}$ is the vector of supplies of the risky assets (Exercise 4.1). Assume $0 < H_U < H$, and set $H_I = H - H_U$.

(a) Show that the equilibrium price of the first asset is $p_1 = p_1^*$, and the equilibrium price of the second asset is

$$p_2 = p_2^* - \frac{\alpha}{HR_f}\left(\frac{H_U}{H_I}\right)\left(\text{var}(\tilde{x}_2) - \frac{\text{cov}(\tilde{x}_1, \tilde{x}_2)^2}{\text{var}(\tilde{x}_1)}\right) < p_2^* \, .$$

Solution: The optimal portfolio of investors who do not invest in the second risky asset is $\theta_U = (\theta_{U1}\ 0)'$, where

$$\theta_{U1} = \frac{\mathsf{E}[\tilde{x}_1] - R_f p_1}{\alpha\,\text{var}(\tilde{x}_1)} \, .$$

The optimal portfolio of investors who invest in both risky assets satisfies

$$\alpha\Sigma\theta_I = \mu - R_f p \, .$$

Let σ_{ij} denote the (i, j)–th element of Σ. The market clearing condition $\bar{\theta} = H_I\theta_I + H_U\theta_U$ implies

$$\begin{aligned}\alpha\Sigma\bar{\theta} = H_I\alpha\Sigma\theta_I + H_U\alpha\Sigma\theta_U &= H_I(\mu - R_f p) + \frac{H_U}{\sigma_{11}}\Sigma\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(\mu - R_f p) \\ &= \begin{pmatrix} H & 0 \\ H_U\sigma_{12}/\sigma_{11} & H_I \end{pmatrix}(\mu - R_f p) \, .\end{aligned}$$