Chapter 5

Mean-Variance Analysis

5.1. Suppose there are two risky assets with means $\mu_1 = 1.08$, $\mu_2 = 1.16$, standard deviations $\sigma_1 = 0.25$, $\sigma_2 = 0.35$, and correlation $\rho = 0.30$. Calculate the GMV portfolio and locate it on Figure 5.1.

Solution: The GMV portfolio is

$$\pi = \frac{1}{\iota'\Sigma^{-1}\iota}\Sigma^{-1}\iota.$$

Substituting

$$\Sigma = \begin{pmatrix} 0.0625 & 0.02625 \\ 0.02625 & 0.1225 \end{pmatrix},$$

we obtain

$$\pi = \begin{pmatrix} 0.7264 \\ 0.2736 \end{pmatrix}.$$

Therefore, the mean and standard deviation of the GMV portfolio are $\mu_{gmv} = \mu'\pi = 1.1019$ and $\sigma_{gmv} = \sqrt{\pi'\Sigma\pi} = 0.2293$. This plots as the point that is furthest to the left on the hyperbola in Figure 5.1.

5.2. Assume there is a risk-free asset. Consider an investor with quadratic utility $-(\tilde{w}-\xi)^2/2$, and no labor income.

(a) Explain why the result of Exercise 2.5 implies that the investor will choose a portfolio on the mean-variance frontier.

Solution: From Exercise 2.5, the optimal portfolio is

$$\phi = \frac{\kappa^2}{1+\kappa^2}(\zeta - w_0 R_f)\Sigma^{-1}(\mu - R_f\iota)\,.$$

This is proportional to $\Sigma^{-1}(\mu - R_f\iota)$ and hence is on the mean-variance frontier.

(b) Under what circumstances will the investor choose a mean-variance efficient portfolio? Explain the economics of the condition you derive.

Solution: The frontier portfolios are scalar multiples of the vector $\Sigma^{-1}(\mu - R_f\iota)$. See (5.15). The positive scalar multiples are efficient (because they have $\mu_{\text{targ}} > R_f$), and the negative scalar multiples are inefficient. Therefore, when $\zeta > w_0R_f$, the optimal portfolio for the quadratic utility investor is on the efficient part of the frontier, and when $\zeta < w_0R_f$, the optimal portfolio is on the inefficient part of the frontier. ζ is the bliss level of wealth for the quadratic utility function. When $\zeta < w_0R_f$, the investor can exceed the bliss level by simply holding the risk-free asset. Thus, higher returns can lower utility, so the investor holds an inefficient portfolio of risky assets.

(c) Re-derive the answer to Part (b) using the orthogonal projection characterization of the quadratic utility investor's optimal portfolio presented in Section **??**.

Solution: Given that there is no labor income, $\tilde{y}_p$ in (3.42) is zero. Also, given that there is a risk-free asset, $\zeta_p = \zeta$ and $\mathsf{E}[\tilde{m}_p\zeta_p] = \zeta\mathsf{E}[\tilde{m}_p] = \zeta/R_f$. Therefore, (3.42) implies

$$\tilde{x} = \zeta - (\zeta/R_f - w_0)\tilde{R}_p\,.$$

The return $\tilde{R}_p$ is on the inefficient part of the frontier, so the return producing $\tilde{x}$ is on the efficient part of the frontier if and onl if $\zeta/R_f - w_0 > 0$.

5.3. Suppose that the risk-free return is equal to the expected return of the GMV portfolio ($R_f = B/C$). Show that there is no tangency portfolio.

Hint: Show there is no δ and λ satisfying

$$\delta\Sigma^{-1}(\mu - R_f\iota) = \lambda\pi_{\text{mu}} + (1-\lambda)\pi_{\text{gmv}}\,.$$

Recall that we are assuming μ is not a scalar multiple of ι.

Solution: The mean-variance frontier considering only the risky assets is the set $\lambda\pi_\mu + (1-\lambda)\pi_\iota$ for some λ, and the mean-variance frontier including the risk-free asset is the set $\delta\Sigma^{-1}(\mu - R_f\iota)$ for some δ. For the frontiers to intersect, we must have

$$\delta\Sigma^{-1}(\mu - R_f\iota) = \lambda\pi_\mu + (1-\lambda)\pi_\iota\,.$$

This is equivalent to

$$\left(\delta - \frac{\lambda}{\iota'\Sigma^{-1}\mu}\right)\Sigma^{-1}\mu = \left(\delta R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota}\right)\Sigma^{-1}\iota\,,$$

and premultiplying by Σ gives

$$\left(\delta - \frac{\lambda}{\iota'\Sigma^{-1}\mu}\right)\mu = \left(\delta R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota}\right)\iota\,.$$

Because μ is not proportional to ι, this equation can hold only if

$$\delta - \frac{\lambda}{\iota'\Sigma^{-1}\mu} = \delta R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota} = 0\,.$$

This implies

$$\frac{\lambda}{\iota'\Sigma^{-1}\mu}R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota} = 0\,,$$

and substituting $R_f = B/C = \iota'\Sigma^{-1}\mu/\iota'\Sigma^{-1}\iota$ yields

$$\frac{1}{\iota'\Sigma^{-1}\iota} = 0\,,$$

which is impossible.

5.4. Show that $\mathsf{E}[\tilde{R}^2] \geq \mathsf{E}[\tilde{R}_p^2]$ for every return $\tilde{R}$ (thus, $\tilde{R}_p$ is the minimum second-moment return). The returns having a given second moment a are the returns satisfying $\mathsf{E}[\tilde{R}^2] = a$, which is equivalent to

$$\text{var}(\tilde{R}) + \mathsf{E}[\tilde{R}]^2 = a\,;$$

thus, they plot on the circle $x^2 + y^2 = a$ in (standard deviation, mean) space. Use the fact that $\tilde{R}_p$ is the minimum second-moment return to illustrate graphically that $\tilde{R}_p$ must be on the inefficient part of the frontier, with and without a risk-free asset (assuming $\mathsf{E}[\tilde{R}_p] > 0$ in the absence of a risk-free asset).

Solution: Using Facts 1, 2 and 8,

$$\mathsf{E}[\tilde{R}^2] = \mathsf{E}[(\tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon})^2] = \mathsf{E}[\tilde{R}_p^2] + b^2\mathsf{E}[\tilde{e}_p^2] + \mathsf{E}[\tilde{\varepsilon}^2] \geq \mathsf{E}[\tilde{R}_p^2]\,.$$

With a risk-free asset, the cone intersects the vertical axis at $R_f > 0$, and the point on the cone closest to the origin is on the lower part. In the absence of a risk-free asset, the assumption $\mathsf{E}[\tilde{R}_p] > 0$ implies that global minimum variance portfolio has a positive expected return (use the definition of b_{m} and Facts 16 and 17 — which imply $1 - \mathsf{E}[\tilde{e}_p] > 0$ — to deduce this). Thus, the point on the hyperbola closest to the origin must be on the lower part of the hyperbola.

5.5. Write any return $\tilde{R}$ as $\tilde{R}_p + (\tilde{R} - \tilde{R}_p)$ and use the fact that $1 - \tilde{e}_p$ is orthogonal to excess returns—because $\tilde{e}_p$ represents the expectation operator on the space of excess returns—to show that

$$\tilde{x} \stackrel{\text{def}}{=} \frac{1}{\mathsf{E}[\tilde{R}_p]}(1 - \tilde{e}_p)$$

is an SDF. When there is a risk-free asset, $\tilde{x}$, being spanned by a constant and an excess return, is in the span of the returns and hence must equal $\tilde{m}_p$. Use this fact to demonstrate (**??**).

Solution: We have

$$\begin{aligned}
\mathsf{E}[\tilde{x}\tilde{R}] &= \frac{1}{\mathsf{E}[\tilde{R}_p]}\mathsf{E}[(1-\tilde{e}_p)\tilde{R}_p] + \frac{1}{\mathsf{E}[\tilde{R}_p]}\mathsf{E}[(1-\tilde{e}_p)(\tilde{R}-\tilde{R}_p)] \\
&= \frac{1}{\mathsf{E}[\tilde{R}_p]}\mathsf{E}[(1-\tilde{e}_p)\tilde{R}_p] \\
&= 1\,,
\end{aligned}$$

using $\mathsf{E}[\tilde{R}-\tilde{R}_p] = \mathsf{E}[\tilde{e}_p(\tilde{R}-\tilde{R}_p]$ for the second equality and Fact 8 for the third. Thus, $\tilde{x}$ is an SDF. This implies

$$\frac{1}{R_f} = \mathsf{E}[\tilde{x}] = \frac{1-\mathsf{E}[\tilde{e}_p]}{\mathsf{E}[\tilde{R}_p]}\,.$$

Moreover, $\tilde{x} = \tilde{m}_p$ implies

$$\tilde{R}_p = \frac{\tilde{x}}{\mathsf{E}[\tilde{x}^2]}\,,$$

and

$$\mathsf{E}[\tilde{x}^2] = \frac{1}{\mathsf{E}[\tilde{R}_p]^2}(1-2\mathsf{E}[\tilde{e}_p]+\mathsf{E}[\tilde{e}_p^2]) = \frac{1-\mathsf{E}[\tilde{e}_p]}{\mathsf{E}[\tilde{R}_p]^2} = \frac{1}{R_f\mathsf{E}[\tilde{R}_p]}\,,$$

using Fact 16 for the second equality. Thus,

$$\tilde{R}_p = R_f\mathsf{E}[\tilde{R}_p]\left(\frac{1}{\mathsf{E}[\tilde{R}_p]}(1-\tilde{e}_p)\right) = R_f(1-\tilde{e}_p)\,.$$

5.6. Establish the properties claimed for the risk-free return proxies:

(a) Show that $\text{var}(\tilde{R}) \geq \text{var}(\tilde{R}_p + b_{\text{m}}\tilde{e}_p)$ for every return $\tilde{R}$.

Solution: By Fact 15, the minimum variance return is $\tilde{R}_p + b\tilde{e}_p$ for some b. Using Fact 8, we have

$$\text{var}(\tilde{R}_p + b\tilde{e}_p) = \text{var}(\tilde{R}_p) - 2b\mathsf{E}[\tilde{R}_p]\mathsf{E}[\tilde{e}_p] + \text{var}(\tilde{e}_p)\,,$$

and by Fact 17, this equals

$$\mathrm{var}(\tilde{R}_p) + \left(b^2(1 - \mathsf{E}[\tilde{e}_p]) - 2b\mathsf{E}[\tilde{R}_p]\right) \mathsf{E}[\tilde{e}_p]\,.$$

By Fact 16, $\mathsf{E}[\tilde{e}_p] > 0$, so the minimum variance return is found by minimizing $(b^2(1-\mathsf{E}[\tilde{e}_p]) - 2b\mathsf{E}[\tilde{R}_p]$ in b, with solution $b = b_{\mathrm{m}}$.

(b) Show that $\mathrm{cov}(\tilde{R}_p, \tilde{R}_p + b_{\mathrm{z}}\tilde{e}_p) = 0$.

Solution: Using Fact 8, we have $\mathrm{cov}(\tilde{R}_p, \tilde{R}_p + b_{\mathrm{z}}\tilde{e}_p) = \mathrm{var}(\tilde{R}_p) - b_{\mathrm{z}}\mathsf{E}[\tilde{R}_p]\mathsf{E}[\tilde{e}_p] = 0$.

(c) Prove (**??**), showing that $\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p$ represents the constant b_c times the expectation operator on the space of returns.

Solution: Using Fact 11 and the definition of b_{c}, we have

$$b_{\mathrm{c}}\mathsf{E}[\tilde{R}] = b_{\mathrm{c}}\mathsf{E}[\tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon}] = \mathsf{E}[\tilde{R}_p^2] + bb_{\mathrm{c}}\mathsf{E}[\tilde{e}_p]\,.$$

From Facts 2, 8, 11, and 16,

$$\begin{aligned}\mathsf{E}[\tilde{R}(\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)] &= \mathsf{E}[(\tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon})(\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)] \\ &= \mathsf{E}[\tilde{R}_p^2] + bb_{\mathrm{c}}\mathsf{E}[\tilde{e}_p^2] \\ &= \mathsf{E}[\tilde{R}_p^2] + bb_{\mathrm{c}}\mathsf{E}[\tilde{e}_p]\,.\end{aligned}$$

Thus,

$$b_{\mathrm{c}}\mathsf{E}[\tilde{R}] = \mathsf{E}[\tilde{R}(\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)]\,.$$

5.7. If all returns are joint normally distributed, then $\tilde{R}_p$, $\tilde{e}_p$ and $\tilde{\varepsilon}$ are joint normally distributed in the orthogonal decomposition $\tilde{R} = \tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon}$ of any return $\tilde{R}$ (because $\tilde{R}_p$ is a return and $\tilde{e}_p$ and $\tilde{\varepsilon}$ are excess returns). Assuming all returns are joint normally distributed, use the orthogonal decomposition to compute the optimal return for a CARA investor.

Solution: When returns are normally distributed, a CARA investor chooses the return $\tilde{R}$ that maximizes

$$\mathsf{E}[\tilde{R}] - \frac{1}{2}\alpha w_0 \operatorname{var}(\tilde{R})\,.$$

Given $\tilde{R} = \tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon}$ and Facts 11 and 15, the objective function is

$$\mathsf{E}[\tilde{R} + b\tilde{e}_p] - \frac{1}{2}\alpha w_0[\operatorname{var}(\tilde{R}_p + b\tilde{e}_p) + \operatorname{var}(\tilde{\varepsilon})]\,,$$

so it is optimal to choose $\tilde{\varepsilon} = 0$. The investor chooses b to maximize

$$b\mathsf{E}[\tilde{e}_p] - \frac{1}{2}\alpha w_0[2b\operatorname{cov}(\tilde{R}_p, \tilde{e}_p) + b^2 \operatorname{var}(\tilde{e}_p)]\,,$$

and the optimum satisfies

$$\mathsf{E}[\tilde{e}_p] - \alpha w_0 \operatorname{cov}(\tilde{R}_p, \tilde{e}_p) - \alpha w_0 \operatorname{var}(\tilde{e}_p) b = 0\,,$$

implying

$$b = \frac{\mathsf{E}[\tilde{e}_p]}{\alpha w_0 \operatorname{var}(\tilde{e}_p)} - \frac{\operatorname{cov}(\tilde{R}_p, \tilde{e}_p)}{\operatorname{var}(\tilde{e}_p)}\,.$$

Using Facts 8 and 17, we can simplify this further to

$$b = \frac{1 + \alpha w_0 \mathsf{E}[\tilde{R}_p]}{\alpha w_0 (1 - \mathsf{E}[\tilde{e}_p])}\,.$$

5.8. Assume there is a risk-free asset.

(a) Using the formula (3.45) for $\tilde{m}_p$, compute λ such that

$$\tilde{R}_p = \lambda \pi_{\text{tang}}' \tilde{\mathbf{R}} + (1 - \lambda) R_f.$$

Solution: We have

$$\tilde{m}_p = \frac{1}{R_f} + \left(\iota - \frac{1}{R_f}\mu\right)' \Sigma^{-1}(\tilde{\mathbf{R}} - \mu)\,.$$

Hence

$$\text{var}(\tilde{m}_p) = \frac{\kappa^2}{R_f^2}\,,$$

where $\kappa^2 = (R_f\iota - \mu)'\Sigma^{-1}(R_f\iota - \mu)$ is the squared maximum Sharpe ratio. Because $\mathsf{E}[\tilde{m}_p] = 1/R_f$, this implies

$$\mathsf{E}[\tilde{m}_p^2] = \frac{1+\kappa^2}{R_f^2}\,.$$

Therefore, by the definition $\tilde{R}_p = \tilde{m}_p/\mathsf{E}[\tilde{m}_p^2]$, we have

$$\tilde{R}_p = \frac{R_f}{1+\kappa^2} + \frac{R_f}{1+\kappa^2}\,(R_f\iota - \mu)'\,\Sigma^{-1}(\tilde{\mathbf{R}} - \mu)\,,$$

in the notation of Section 5.2. Setting

$$\lambda = -\frac{R_f(B - R_f C)}{1+\kappa^2}\,,$$

we have

$$1 - \lambda = \frac{1+\kappa^2 + R_f B - R_f^2 C}{1+\kappa^2} = \frac{1 + A - R_f B}{1+\kappa^2}\,,$$

because $\kappa^2 = A - 2R_f B + R_f^2 C$. Thus,

$$\tilde{R}_p = \lambda\pi_{\text{tang}}'\tilde{\mathbf{R}} + (1-\lambda)R_f\,.$$

(b) Show that λ in part (a) is negative when $R_f < B/C$ and positive when $R_f > B/C$. Note: This shows that $\tilde{R}_p$ is on the inefficient part of the frontier, because the portfolio generating $\tilde{R}_p$ is short the tangency portfolio when the tangency portfolio is efficient and long the tangency portfolio when it is inefficient.

Solution:

$$\lambda = -\frac{R_f(B - R_f C)}{1+\kappa^2} < 0$$

when $B > R_f C$ and positive when $B < R_f C$.

5.9. Consider the problem of choosing a portfolio π of risky assets, a proportion $\phi_b \geq 0$ to borrow and a proportion $\phi_\ell \geq 0$ to lend to maximize the expected return $\pi'\mu + \phi_\ell R_\ell - \phi_b R_b$ subject to the constraints $(1/2)\pi'\Sigma\pi \leq k$ and $\iota'\pi + \phi_\ell - \phi_b = 1$. Assume $B/C > R_b > R_\ell$, where B and C are defined in (**??**). Define

$$\pi_b = \frac{1}{\iota'\Sigma^{-1}(\mu - R_b\iota)}\Sigma^{-1}(\mu - R_b\iota)\,,$$
$$\pi_\ell = \frac{1}{\iota'\Sigma^{-1}(\mu - R_\ell\iota)}\Sigma^{-1}(\mu - R_\ell\iota)\,.$$

Using the Kuhn-Tucker conditions, show that the solution is either (i) $\pi = (1-\phi_\ell)\pi_\ell$ for $0 \leq \phi_\ell \leq 1$, (ii) $\pi = \lambda\pi_\ell + (1-\lambda)\pi_b$ for $0 \leq \lambda \leq 1$, or (iii) $\pi = (1+\phi_b)\pi_b$ for $\phi_b \geq 0$.

Solution: The Kuhn-Tucker conditions are

$$\begin{aligned}
\mu - \delta\Sigma\pi - \gamma\iota &= 0\,,\\
R_\ell - \gamma + \eta_\ell &= 0\,,\\
-R_b + \gamma + \eta_b &= 0\,,\\
\phi_\ell, \phi_b, \eta_\ell, \eta_b, \delta &\geq 0\,,\\
\frac{1}{2}\pi'\Sigma\pi &\leq k\,,\\
\iota'\pi + \phi_\ell - \phi_b &= 1\,,\\
\eta_\ell\phi_\ell = \eta_b\phi_b = \delta\left(\frac{1}{2}\pi'\Sigma\pi - k\right) &= 0\,.
\end{aligned}$$

There are three possibilities to consider: (i) $\phi_\ell > 0$, (ii), $\phi_b > 0$, (iii) $\phi_\ell = \phi_b = 0$.

(i) If $\phi_\ell > 0$, then $\eta_\ell = 0$, $\gamma = R_\ell$, and

$$\pi = \frac{1}{\delta}\Sigma^{-1}(\mu - R_\ell\iota)\,.$$

Also, $\gamma = R_\ell$ implies $\eta_b = R_b - R_\ell > 0$. Hence, $\phi_b = 0$, and $\iota'\pi = 1 - \phi_\ell$. This implies $\pi = (1-\phi_\ell)\pi_\ell$.

(ii) If $\phi_b > 0$, then $\eta_b = 0$, $\gamma = -R_b$, and

$$\pi = \frac{1}{\delta}\Sigma^{-1}(\mu - R_b\iota)\,.$$

Also, $\gamma = -R_b$ implies $\eta_\ell = R_b - R_\ell > 0$, so $\phi_\ell = 0$. This implies $\iota'\pi = 1 + \phi_b$. Hence, $\pi = (1+\phi_b)\pi_b$.

(iii) If $\phi_\ell = \phi_b = 0$, then

$$\pi = \frac{1}{\delta}\Sigma^{-1}(\mu - \gamma\iota)\,,$$

where $\gamma = R_\ell + \eta_\ell \geq R_\ell$ and $\gamma = R_b - \eta_b \leq R_b$. Thus, $\gamma = \lambda R_\ell + (1-\lambda)R_b$ for some $0 \leq \lambda \leq 1$. From $\iota'\pi = 1$, it follows that $\pi = \lambda\pi_\ell + (1-\lambda)\pi_b$.