Solution: From (4.4),

$$\lambda_h u'_{h1}(\tilde{c}_{h1}) = \tilde{\eta}_1$$

for all h. By (3.9),

$$\frac{u'_{h1}(\tilde{c}_{h1})}{u'_{h0}(c_{h0})}$$

is an SDF for each h. Combining these, we have that

$$\frac{\tilde{\eta}_1}{\lambda_h u'_{h0}(c_{h0})}$$

is an SDF for each h. Thus,

$$\mathsf{E}\left[\frac{\tilde{\eta}_1}{\lambda_h u'_{h0}(c_{h0})}\right] = \frac{1}{R_f}\,.$$

This implies

$$\lambda_h u'_{h0}(\tilde{c}_{h0}) = \eta_0\,,$$

where we define $\eta_0 = R_f \mathsf{E}[\tilde{\eta}_1]$. Thus, the first-order conditions for the social planner's problem are satisfied at both date 0 and date 1. By concavity, this implies Pareto optimality.

4.9. Consider a model with date–0 endowments y_{h0} and date–0 consumption c_{h0}. Suppose all investors have log utility, a common discount factor δ, and no date–1 endowments. Do not assume markets are complete. Show that, in a competitive equilibrium, the date–0 value of the market portfolio is $\delta \sum_{h=1}^{H} c_{h0}$.

Solution: The marginal rate of substitution for investor h is $\delta c_{h0}/\tilde{c}_{h1}$. This is an SDF, so the cost of the date 1 consumption of investor h is

$$\mathsf{E}\left[\frac{\delta c_{h0}}{\tilde{c}_{h1}}\tilde{c}_{h1}\right] = \delta c_{h0}\,.$$

Adding over investors, the total cost of date 1 consumption (that is, the cost of the market portfolio) is $\delta \sum_{h=1}^{H} c_{h0}$.

4.10. Suppose the payoff of the market portfolio $\tilde{w}_m$ has k possible values. Denote these possible values by $a_1 < \cdots < a_k$. For convenience, suppose $a_i - a_{i-1}$ is the same number Δ for each i. Suppose there is a risk-free asset with payoff equal to 1. Suppose there are $k-1$ call options on the market portfolio, with the exercise price of the ith option being a_i. The payoff of the ith option is $\max(0, \tilde{w}_m - a_i)$.

(a) Show for each $i = 1, \ldots, k-2$ that a portfolio that is long one unit of option i and short one unit of option $i+1$ pays Δ if $\tilde{w}_m \geq a_{i+1}$ and 0 otherwise. (This portfolio of options is a bull spread.)

Solution: When $\tilde{w}_m \geq a_{i+1}$, a long position in option i pays $\tilde{w}_m - a_i$, and a short position in option $i+1$ has cash flow $a_{i+1} - \tilde{w}_m$. The sum of these is $a_{i+1} - a_i = \varDelta$.

(b) Consider the following k portfolios. Show that the payoff of portfolio i is 1 when $\tilde{w}_m = a_i$ and 0 otherwise. (Thus, these are Arrow securities for the events on which $\tilde{w}_m$ is constant.)

(i) $i = 1$: long one unit of the risk-free asset, short $1/\Delta$ units of option 1, and long $1/\Delta$ units of option 2. (This portfolio of options is a short bull spread.)

(ii) $1 < i < k$: long $1/\Delta$ units of option $i-1$, short $2/\Delta$ units of option i, and long $1/\Delta$ units of option $i+1$. (These portfolios are butterfly spreads.)

(iii) $i = k-1$: long $1/\Delta$ units of option $k-2$ and short $2/\Delta$ units of option $k-1$.

(iv) $i = k$: long $1/\Delta$ units of option $k-1$.

Solution:

(i) Being long $1/\varDelta$ units of option 2 and short $1/\varDelta$ units of option 1 pays -1 when $\tilde{w}_m \geq a_2$ and 0 otherwise. Combining this with a payoff of 1 yields 1 when $\tilde{w}_m = a_1$ and 0 otherwise.

(ii) Being long $1/\Delta$ units of option $i-1$ and short $1/\Delta$ units of option i pays 1 when $\tilde{w}_m \geq a_i$ and 0 otherwise. Being short $1/\Delta$ units of option i and long $1/\Delta$ units of option $i+1$ pays -1 when $\tilde{w}_m \geq a_{i+1}$ and 0 otherwise. The sum of these pays 1 when $\tilde{w}_m = a_i$ and 0 otherwise.

(iii) Being long $1/\Delta$ units of option $k-2$ produces a payoff of 1 when $\tilde{w}_m = a_{k-1}$ and 2 when $\tilde{w}_m = a_k$. Being short $2/\Delta$ units of option $k-1$ produces a payoff of -2 when $\tilde{w}_m = a_k$. The sum of the two pays 1 when $\tilde{w}_m = a_{k-1}$ and 0 otherwise.

(iv) Being long $1/\Delta$ units of option $k-1$ produces a payoff of 1 when $\tilde{w}_m = a_k$ and 0 otherwise.

(c) Given any function f, define $\tilde{z} = f(\tilde{w}_m)$. Show that there is a portfolio of the risk-free asset and the call options with payoff equal to $\tilde{z}$.

Solution: Let z_i denote the value of $\tilde{z}$ when $\tilde{w}_m = a_i$. The portfolio consisting of z_1 units of the risk-free asset, $(z_2 - z_1)/\Delta$ units of option 1, and $(z_{i+1} - 2z_i + z_{i-1})/\Delta$ units of option i for $i = 2, \ldots, k-1$ has payoff equal to $\tilde{z}$.

Chapter 5

Mean-Variance Analysis

5.1. Suppose there are two risky assets with means $\mu_1 = 1.08$, $\mu_2 = 1.16$, standard deviations $\sigma_1 = 0.25$, $\sigma_2 = 0.35$, and correlation $\rho = 0.30$. Calculate the GMV portfolio and locate it on Figure 5.1.

Solution: The GMV portfolio is

$$\pi = \frac{1}{\iota'\Sigma^{-1}\iota}\Sigma^{-1}\iota.$$

Substituting

$$\Sigma = \begin{pmatrix} 0.0625 & 0.02625 \\ 0.02625 & 0.1225 \end{pmatrix},$$

we obtain

$$\pi = \begin{pmatrix} 0.7264 \\ 0.2736 \end{pmatrix}.$$

Therefore, the mean and standard deviation of the GMV portfolio are $\mu_{gmv} = \mu'\pi = 1.1019$ and $\sigma_{gmv} = \sqrt{\pi'\Sigma\pi} = 0.2293$. This plots as the point that is furthest to the left on the hyperbola in Figure 5.1.

5.2. Assume there is a risk-free asset. Consider an investor with quadratic utility $-(\tilde{w}-\xi)^2/2$, and no labor income.

(a) Explain why the result of Exercise 2.5 implies that the investor will choose a portfolio on the mean-variance frontier.

Solution: From Exercise 2.5, the optimal portfolio is

$$\phi = \frac{\kappa^2}{1+\kappa^2}(\zeta - w_0 R_f)\Sigma^{-1}(\mu - R_f\iota)\,.$$

This is proportional to $\Sigma^{-1}(\mu - R_f\iota)$ and hence is on the mean-variance frontier.

(b) Under what circumstances will the investor choose a mean-variance efficient portfolio? Explain the economics of the condition you derive.

Solution: The frontier portfolios are scalar multiples of the vector $\Sigma^{-1}(\mu - R_f\iota)$. See (5.15). The positive scalar multiples are efficient (because they have $\mu_{\text{targ}} > R_f$), and the negative scalar multiples are inefficient. Therefore, when $\zeta > w_0 R_f$, the optimal portfolio for the quadratic utility investor is on the efficient part of the frontier, and when $\zeta < w_0 R_f$, the optimal portfolio is on the inefficient part of the frontier. ζ is the bliss level of wealth for the quadratic utility function. When $\zeta < w_0 R_f$, the investor can exceed the bliss level by simply holding the risk-free asset. Thus, higher returns can lower utility, so the investor holds an inefficient portfolio of risky assets.

(c) Re-derive the answer to Part (b) using the orthogonal projection characterization of the quadratic utility investor's optimal portfolio presented in Section **??**.

Solution: Given that there is no labor income, $\tilde{y}_p$ in (3.42) is zero. Also, given that there is a risk-free asset, $\zeta_p = \zeta$ and $\mathsf{E}[\tilde{m}_p\zeta_p] = \zeta\mathsf{E}[\tilde{m}_p] = \zeta/R_f$. Therefore, (3.42) implies

$$\tilde{x} = \zeta - (\zeta/R_f - w_0)\tilde{R}_p\,.$$

The return $\tilde{R}_p$ is on the inefficient part of the frontier, so the return producing $\tilde{x}$ is on the efficient part of the frontier if and onl if $\zeta/R_f - w_0 > 0$.

5.3. Suppose that the risk-free return is equal to the expected return of the GMV portfolio ($R_f = B/C$). Show that there is no tangency portfolio.

Hint: Show there is no δ and λ satisfying

$$\delta\Sigma^{-1}(\mu - R_f\iota) = \lambda\pi_{\text{mu}} + (1-\lambda)\pi_{\text{gmv}}\,.$$

Recall that we are assuming μ is not a scalar multiple of ι.

Solution: The mean-variance frontier considering only the risky assets is the set $\lambda\pi_\mu + (1-\lambda)\pi_\iota$ for some λ, and the mean-variance frontier including the risk-free asset is the set $\delta\Sigma^{-1}(\mu - R_f\iota)$ for some δ. For the frontiers to intersect, we must have

$$\delta\Sigma^{-1}(\mu - R_f\iota) = \lambda\pi_\mu + (1-\lambda)\pi_\iota\,.$$

This is equivalent to

$$\left(\delta - \frac{\lambda}{\iota'\Sigma^{-1}\mu}\right)\Sigma^{-1}\mu = \left(\delta R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota}\right)\Sigma^{-1}\iota\,,$$

and premultiplying by Σ gives

$$\left(\delta - \frac{\lambda}{\iota'\Sigma^{-1}\mu}\right)\mu = \left(\delta R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota}\right)\iota\,.$$

Because μ is not proportional to ι, this equation can hold only if

$$\delta - \frac{\lambda}{\iota'\Sigma^{-1}\mu} = \delta R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota} = 0\,.$$

This implies

$$\frac{\lambda}{\iota'\Sigma^{-1}\mu}R_f + \frac{1-\lambda}{\iota'\Sigma^{-1}\iota} = 0\,,$$

and substituting $R_f = B/C = \iota'\Sigma^{-1}\mu/\iota'\Sigma^{-1}\iota$ yields

$$\frac{1}{\iota'\Sigma^{-1}\iota} = 0\,,$$

which is impossible.

5.4. Show that $\mathsf{E}[\tilde{R}^2] \geq \mathsf{E}[\tilde{R}_p^2]$ for every return $\tilde{R}$ (thus, $\tilde{R}_p$ is the minimum second-moment return). The returns having a given second moment a are the returns satisfying $\mathsf{E}[\tilde{R}^2] = a$, which is equivalent to

$$\text{var}(\tilde{R}) + \mathsf{E}[\tilde{R}]^2 = a\,;$$

thus, they plot on the circle $x^2 + y^2 = a$ in (standard deviation, mean) space. Use the fact that $\tilde{R}_p$ is the minimum second-moment return to illustrate graphically that $\tilde{R}_p$ must be on the inefficient part of the frontier, with and without a risk-free asset (assuming $\mathsf{E}[\tilde{R}_p] > 0$ in the absence of a risk-free asset).

Solution: Using Facts 1, 2 and 8,

$$\mathsf{E}[\tilde{R}^2] = \mathsf{E}[(\tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon})^2] = \mathsf{E}[\tilde{R}_p^2] + b^2\mathsf{E}[\tilde{e}_p^2] + \mathsf{E}[\tilde{\varepsilon}^2] \geq \mathsf{E}[\tilde{R}_p^2]\,.$$

With a risk-free asset, the cone intersects the vertical axis at $R_f > 0$, and the point on the cone closest to the origin is on the lower part. In the absence of a risk-free asset, the assumption $\mathsf{E}[\tilde{R}_p] > 0$ implies that global minimum variance portfolio has a positive expected return (use the definition of b_{m} and Facts 16 and 17 — which imply $1 - \mathsf{E}[\tilde{e}_p] > 0$ — to deduce this). Thus, the point on the hyperbola closest to the origin must be on the lower part of the hyperbola.

5.5. Write any return $\tilde{R}$ as $\tilde{R}_p + (\tilde{R} - \tilde{R}_p)$ and use the fact that $1 - \tilde{e}_p$ is orthogonal to excess returns—because $\tilde{e}_p$ represents the expectation operator on the space of excess returns—to show that

$$\tilde{x} \stackrel{\text{def}}{=} \frac{1}{\mathsf{E}[\tilde{R}_p]}(1 - \tilde{e}_p)$$

is an SDF. When there is a risk-free asset, $\tilde{x}$, being spanned by a constant and an excess return, is in the span of the returns and hence must equal $\tilde{m}_p$. Use this fact to demonstrate (**??**).

Solution: We have

$$
\begin{aligned}
\mathsf{E}[\tilde{x}\tilde{R}] &= \frac{1}{\mathsf{E}[\tilde{R}_p]}\mathsf{E}[(1-\tilde{e}_p)\tilde{R}_p] + \frac{1}{\mathsf{E}[\tilde{R}_p]}\mathsf{E}[(1-\tilde{e}_p)(\tilde{R}-\tilde{R}_p)] \\
&= \frac{1}{\mathsf{E}[\tilde{R}_p]}\mathsf{E}[(1-\tilde{e}_p)\tilde{R}_p] \\
&= 1\,,
\end{aligned}
$$

using $\mathsf{E}[\tilde{R}-\tilde{R}_p] = \mathsf{E}[\tilde{e}_p(\tilde{R}-\tilde{R}_p]$ for the second equality and Fact 8 for the third. Thus, $\tilde{x}$ is an SDF. This implies

$$
\frac{1}{R_f} = \mathsf{E}[\tilde{x}] = \frac{1-\mathsf{E}[\tilde{e}_p]}{\mathsf{E}[\tilde{R}_p]}\,.
$$

Moreover, $\tilde{x} = \tilde{m}_p$ implies

$$
\tilde{R}_p = \frac{\tilde{x}}{\mathsf{E}[\tilde{x}^2]}\,,
$$

and

$$
\mathsf{E}[\tilde{x}^2] = \frac{1}{\mathsf{E}[\tilde{R}_p]^2}(1-2\mathsf{E}[\tilde{e}_p]+\mathsf{E}[\tilde{e}_p^2]) = \frac{1-\mathsf{E}[\tilde{e}_p]}{\mathsf{E}[\tilde{R}_p]^2} = \frac{1}{R_f\mathsf{E}[\tilde{R}_p]}\,,
$$

using Fact 16 for the second equality. Thus,

$$
\tilde{R}_p = R_f\mathsf{E}[\tilde{R}_p]\left(\frac{1}{\mathsf{E}[\tilde{R}_p]}(1-\tilde{e}_p)\right) = R_f(1-\tilde{e}_p)\,.
$$

5.6. Establish the properties claimed for the risk-free return proxies:

(a) Show that $\text{var}(\tilde{R}) \geq \text{var}(\tilde{R}_p + b_{\text{m}}\tilde{e}_p)$ for every return $\tilde{R}$.

Solution: By Fact 15, the minimum variance return is $\tilde{R}_p + b\tilde{e}_p$ for some b. Using Fact 8, we have

$$
\text{var}(\tilde{R}_p + b\tilde{e}_p) = \text{var}(\tilde{R}_p) - 2b\mathsf{E}[\tilde{R}_p]\mathsf{E}[\tilde{e}_p] + \text{var}(\tilde{e}_p)\,,
$$

and by Fact 17, this equals

$$\mathrm{var}(\tilde{R}_p) + \left(b^2(1 - \mathsf{E}[\tilde{e}_p]) - 2b\mathsf{E}[\tilde{R}_p]\right) \mathsf{E}[\tilde{e}_p]\,.$$

By Fact 16, $\mathsf{E}[\tilde{e}_p] > 0$, so the minimum variance return is found by minimizing $(b^2(1-\mathsf{E}[\tilde{e}_p]) - 2b\mathsf{E}[\tilde{R}_p]$ in b, with solution $b = b_{\mathrm{m}}$.

(b) Show that $\mathrm{cov}(\tilde{R}_p, \tilde{R}_p + b_{\mathrm{z}}\tilde{e}_p) = 0$.

Solution: Using Fact 8, we have $\mathrm{cov}(\tilde{R}_p, \tilde{R}_p + b_{\mathrm{z}}\tilde{e}_p) = \mathrm{var}(\tilde{R}_p) - b_{\mathrm{z}}\mathsf{E}[\tilde{R}_p]\mathsf{E}[\tilde{e}_p] = 0$.

(c) Prove (**??**), showing that $\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p$ represents the constant b_c times the expectation operator on the space of returns.

Solution: Using Fact 11 and the definition of b_{c}, we have

$$b_{\mathrm{c}}\mathsf{E}[\tilde{R}] = b_{\mathrm{c}}\mathsf{E}[\tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon}] = \mathsf{E}[\tilde{R}_p^2] + bb_{\mathrm{c}}\mathsf{E}[\tilde{e}_p]\,.$$

From Facts 2, 8, 11, and 16,

$$\begin{aligned}\mathsf{E}[\tilde{R}(\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)] &= \mathsf{E}[(\tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon})(\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)] \\ &= \mathsf{E}[\tilde{R}_p^2] + bb_{\mathrm{c}}\mathsf{E}[\tilde{e}_p^2] \\ &= \mathsf{E}[\tilde{R}_p^2] + bb_{\mathrm{c}}\mathsf{E}[\tilde{e}_p]\,.\end{aligned}$$

Thus,

$$b_{\mathrm{c}}\mathsf{E}[\tilde{R}] = \mathsf{E}[\tilde{R}(\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)]\,.$$

5.7. If all returns are joint normally distributed, then $\tilde{R}_p$, $\tilde{e}_p$ and $\tilde{\varepsilon}$ are joint normally distributed in the orthogonal decomposition $\tilde{R} = \tilde{R}_p + b\tilde{e}_p + \tilde{\varepsilon}$ of any return $\tilde{R}$ (because $\tilde{R}_p$ is a return and $\tilde{e}_p$ and $\tilde{\varepsilon}$ are excess returns). Assuming all returns are joint normally distributed, use the orthogonal decomposition to compute the optimal return for a CARA investor.