3.4. Suppose two random vectors $\tilde{X}$ and $\tilde{Y}$ are joint normally distributed. Explain why the orthogonal projection (3.32) equals $\mathsf{E}[\tilde{Y}|\tilde{X}]$.

Solution: Let $\tilde{Y}_p$ denote the projection (3.32), so we have $\tilde{Y} = \tilde{Y}_p + \tilde{\varepsilon}$ with $\tilde{Y}_p$ being an affine function of $\tilde{X}$ and $\tilde{\varepsilon}$ being orthogonal to $\tilde{X}$. Then,

$$\mathsf{E}[\tilde{Y} \mid \tilde{X}] = \mathsf{E}[\tilde{Y}_p \mid \tilde{X}] + \mathsf{E}[\tilde{\varepsilon} \mid \tilde{X}] = \tilde{Y}_p + \mathsf{E}[\tilde{\varepsilon} \mid \tilde{X}]\,.$$

Now, because $\tilde{\varepsilon} = \tilde{Y} - \tilde{Y}_p$, which is a linear combination of the joint normal random vectors $\tilde{X}$ and $\tilde{Y}$, it follows that $\tilde{\varepsilon}$ and $\tilde{X}$ are joint normal. Hence, because they are uncorrelated, they are actually independent and consequently mean-independent. This implies that $\mathsf{E}[\tilde{\varepsilon} \mid \tilde{X}] = 0$, so

$$\mathsf{E}[\tilde{Y} \mid \tilde{X}] = \tilde{Y}_p\,.$$

3.5. Show that, if there is a strictly positive SDF, then there are no arbitrage opportunities.

Solution: Assume $\tilde{m}$ is a strictly positive SDF. If $\tilde{x}$ is a nonnegative marketed payoff, then its price is $\mathsf{E}[\tilde{m}\tilde{x}] \geq 0$, and $\mathsf{E}[\tilde{m}\tilde{x}] = 0$ if and only if $\tilde{x} = 0$ with probability one. Therefore, there are no arbitrage opportunities.

3.6. Show by example that the law of one price can hold but there can still be arbitrage opportunities.

Solution: Suppose there are two possible states of the world, and the market consists of the two Arrow securities having prices p_i. Then the market is complete, and each payoff $\tilde{x} = (x_1, x_2)$ has a unique cost $p_1x_1 + p_2x_2$. If $p_1 < 0$, then buying the first asset is an arbitrage opportunity.

3.7. Suppose there is an SDF $\tilde{m}$ with the property that for every function g there exists a portfolio θ (depending on g) such that

$$\sum_{i=1}^{n} \theta_i \tilde{x}_i = g(\tilde{m})\,.$$

Consider an investor with no labor income $\tilde{y}$. Show that his optimal wealth is a function of $\tilde{m}$. Hint: For any feasible $\tilde{w}$, define $\tilde{w}^* = \mathsf{E}[\tilde{w} \mid \tilde{m}]$, and show that $\tilde{w}^*$ is both budget feasible and at least as preferred as $\tilde{w}$, using the result of Section 1.5. Note: The assumption in this exercise is a weak form of market completeness. The exercise is inspired by Chamberlain (1988).

Solution: Set $\tilde{w}^* = \mathsf{E}[\tilde{w}|\tilde{m}]$ and $\tilde{\varepsilon} = \tilde{w} - \tilde{w}^*$, so we have that $\tilde{w}$ is $\tilde{w} = \tilde{w}^* + \tilde{\varepsilon}$. We will show that $\tilde{\varepsilon}$ has a zero mean and is mean-independent of $\tilde{w}^*$. Hence, the result of Section 1.5 shows that $\tilde{w}^*$ is at least as preferred as $\tilde{w}$. Finally, we will show that $\tilde{w}^*$ is budget feasible. This implies that $\tilde{w}^*$ is optimal. Since $\tilde{w}^* = \mathsf{E}[\tilde{w}|\tilde{m}]$, which is a function of $\tilde{m}$, this will complete the proof.

We have

$$\mathsf{E}[\tilde{\varepsilon} \mid \tilde{m}] = \mathsf{E}[\tilde{w} \mid \tilde{m}] - \mathsf{E}[\tilde{w}^* \mid \tilde{m}] = \tilde{w}^* - \tilde{w}^* = 0\,.$$

Also, because $\tilde{w}^*$ is a function of $\tilde{m}$,

$$\mathsf{E}[\tilde{\varepsilon} \mid \tilde{w}^*] = \mathsf{E}\left[\mathsf{E}[\tilde{\varepsilon} \mid \tilde{m}] \mid \tilde{w}^*\right] = 0\,.$$

Therefore, $\tilde{\varepsilon}$ has a zero mean and is mean-independent of $\tilde{w}^*$. Because $\tilde{w}^*$ is a function of $\tilde{m}$, there exists by assumption a portfolio $\tilde{\theta}$ with payoff equal to $\tilde{w}^*$. The cost of the portfolio is

$$\mathsf{E}[\tilde{m}\tilde{w}] = \mathsf{E}[\mathsf{E}[\tilde{m}\tilde{w} \mid \tilde{m}]] = \mathsf{E}[\tilde{m}\mathsf{E}[\tilde{w} \mid \tilde{m}]] = \mathsf{E}[\tilde{m}\tilde{w}^*]\,,$$

by iterated expectations. Hence, the cost of $\tilde{w}^*$ is the same as the cost of $\tilde{w}$, so $\tilde{w}^*$ is budget feasible.

3.8. Suppose there is a risk-free asset. Adopt the notation of Exercise 3.7, and assume the risky asset returns have a joint normal distribution. Show that the optimal portfolio of risky assets for an investor with no labor income is $\pi = \delta\Sigma^{-1}(\mu - R_f\iota)$ for some real number δ, by applying the reasoning of Exercise 3.7 with $\tilde{m} = \tilde{m}_p$, using the formula (3.45) for $\tilde{m}_p$ and using the results of Exercise 3.4.

Solution: For any budget feasible $\tilde{w}$, let $\tilde{w}^* = \mathsf{E}[\tilde{w} \mid \tilde{m}_p]$. Then, as shown in Exercise 3.7, $\tilde{w}$ equals $\tilde{w}^*$ plus mean-independent noise, so $\tilde{w}^*$ is preferred to $\tilde{w}$. Furthermore, $\tilde{w}^*$ is budget feasible. From (3.45),

$$\tilde{m}_p - \mathsf{E}[\tilde{m}_p] = -\frac{1}{R_f}(\mu - R_f\iota)'\Sigma^{-1}(\tilde{R}^{\text{vec}} - \mu)\,.$$

Hence,

$$\tilde{w}^* = \mathsf{E}[\tilde{w}] - \frac{1}{R_f}\left(\frac{\text{cov}(\tilde{w}, \tilde{m}_p)}{\text{var}(\tilde{m}_p)}\right)(\mu - R_f\iota)'\Sigma^{-1}(\tilde{R}^{\text{vec}} - \mu)\,.$$

This shows that the portfolio of risky assets producing $\tilde{w}^*$ is $\delta\Sigma^{-1}(\mu - R_f\iota)$ for

$$\delta = -\frac{1}{R_f}\left(\frac{\text{cov}(\tilde{w}, \tilde{m}_p)}{\text{var}(\tilde{m}_p)}\right) = -\frac{1}{R_f}\left(\frac{\text{cov}(\tilde{w}^*, \tilde{m}_p)}{\text{var}(\tilde{m}_p)}\right)\,,$$

the second equality following from iterated expectations.

3.9. Assume there is a finite number of assets, and the payoff of each asset has a finite variance. Assume the Law of One Price holds. Apply facts stated in Section 3.8 to show that there is a unique SDF $\tilde{m}_p$ in the span of the asset payoffs. Show that the orthogonal projection of any other SDF onto the span of the asset payoffs equals $\tilde{m}_p$.

Solution: The span of the assets is a finite-dimensional subspace of $\mathcal{L}^2$. The law of one price states that there is a unique price $C[\tilde{x}]$ for each $\tilde{x}$ in the span of the payoffs. The function $C[\cdot]$ is linear. Therefore, it has a Riesz representation $C[\tilde{x}] = \mathsf{E}[\tilde{x}\tilde{m}_p]$ for a unique $\tilde{m}_p$ in the span of the assets. Given any stochastic discount factor $\tilde{m}$, we have $\tilde{m} = \tilde{m}^* + \tilde{\varepsilon}$, where the orthogonal projection $\tilde{m}^*$ is in the span of the assets and $\tilde{\varepsilon}$ is orthogonal to the span of the assets. Hence, $C[\tilde{x}] = \mathsf{E}[\tilde{m}\tilde{x}] = \mathsf{E}[\tilde{x}\tilde{m}^*]$ for all $\tilde{x}$ in the span of the assets. Thus, $\tilde{m}^*$ is also in the span of the assets and represents the price function. By the uniqueness of the Riesz representation, it must be that $\tilde{m}^* = \tilde{m}_p$.

Chapter 4

Equilibrium and Efficiency

4.1. Suppose there are n risky assets with normally distributed payoffs $\tilde{x}_i$. Assume all investors have CARA utility and no labor income $\tilde{y}_h$. Define α to be the aggregate absolute risk aversion as in Section 1.1. Assume there is a risk-free asset in zero net supply. Let $\bar{\theta} = (\bar{\theta}_1 \cdots \bar{\theta}_n)'$ denote the vector of supplies of the n risky assets. Let μ denote the mean and Σ the covariance matrix of the vector $\tilde{X} = (\tilde{x}_1 \cdots \tilde{x}_n)'$ of asset payoffs. Assume Σ is nonsingular. Suppose the utility functions of investor h are

$$u_0(c) = -e^{-\alpha_h c}$$

and

$$u_1(c) = -\delta_h e^{-\alpha_h c}$$

Let $\bar{c}_0$ denote the aggregate endowment $\sum_{h=1}^{H} y_{h0}$ at date 0.

(a) Use the results of Exercise 2.2 on the optimal demands for the risky assets to show that the equilibrium price vector is

$$p = \frac{1}{R_f}\left(\mu - \alpha\Sigma\bar{\theta}\right)$$

Solution: From Exercise 2.2, the optimal portfolio of investor h is

$$\theta_h = \frac{1}{\alpha_h}\Sigma^{-1}(\mu - R_f p)\,.$$

Thus, the aggregate demand for risky assets is

$$\left(\sum_{h=1}^{H}\frac{1}{\alpha_h}\right)\Sigma^{-1}(\mu - R_f p) = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f p)\,.$$

Market clearing implies

$$\bar\theta = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f p) \quad \Rightarrow \quad p = \frac{1}{R_f}(\mu - \alpha\Sigma\bar\theta)\,.$$

(b) Interpret the risk adjustment vector $\alpha\Sigma\bar\theta$ in (4.24), explaining in economic terms why a large element of this vector implies an asset has a low price relative to its expected payoff.

Solution: The formula states that equilibrium prices equal expected payoffs discounted at the risk-free rate minus a penalty for risk. The penalty for risk depends on risk aversion α, the covariance matrix Σ of asset payoffs, and the number of shares outstanding (the vector $\bar\theta$). Roughly speaking, it states that prices are lower when investors are more risk averse or when there is more risk in the supplies of assets. The ith element of the vector $\Sigma\bar\theta$ is the sum (for $j = 1, \ldots, n$) of the covariance of asset i with asset j multiplied by the number of shares outstanding of asset j. Thus, the vector $\Sigma\bar\theta$ is the vector of covariances of the n assets with the market portfolio $\bar\theta$. So, more precisely, the formula states that the penalty for risk for each asset i is aggregate risk aversion times the covariance of asset i with the market portfolio. That covariance determines how much risk asset i adds at the margin for an investor holding the market portfolio (we will see more on this in Chapters 5 and 6).

(c) Assume δ_h is the same for all h (denote the common value by δ). Use the market-clearing condition for the date-0 consumption good to deduce that the equilibrium risk-free return is

$$R_f = \frac{1}{\delta} exp\left(\alpha\left(\bar{\theta}'\mu - c_0\right) - \frac{1}{2}\alpha^2\bar{\theta}'\Sigma\bar{\theta}\right)$$

Solution: To derive R_f we need to clear the market for date–0 consumption (or the market for the risk-free asset). Investor h chooses c_0 and θ to maximize

$$\begin{aligned} &-\exp(-\alpha_h c_0) - \delta_h \mathsf{E}\Big[\exp\big(-\alpha_h[(w_{h0} - c_0 - p'\theta)R_f + \theta'\tilde{x}]\big)\Big] \\ &\qquad = -\exp(-\alpha_h c_0) - \delta_h \exp\big(-\alpha_h(w_{h0} - c_0 - p'\theta)R_f\big)\mathsf{E}\Big[\exp\big(-\alpha_h\theta'\tilde{x}\big)\Big]\,. \end{aligned}$$

Substituting the optimal portfolio $\theta = \theta_h$ yields

$$\mathsf{E}\Big[\exp\big(-\alpha_h\theta'\tilde{x}\big)\Big] = \exp\left(-(\mu - R_f p)'\Sigma^{-1}\mu + \frac{1}{2}(\mu - R_f p)'\Sigma^{-1}(\mu - R_f p)\right)\,,$$

and substituting $p = \frac{1}{R_f}(\mu - \alpha\Sigma\bar{\theta})$ yields

$$\mathsf{E}\Big[\exp\big(-\alpha_h\theta'\tilde{x}\big)\Big] = \exp\left(-\alpha\bar{\theta}'\mu + \frac{\alpha^2}{2}\bar{\theta}'\Sigma\theta\right)\,,$$

Thus, the first-order condition for c_{h0} is

$$\alpha_h\exp(-\alpha_h c_{h0}) = \delta_h\alpha_h R_f \exp\big(-\alpha_h(w_{h0} - c_{h0} - p'\theta_h)R_f\big)\exp\left(-\alpha\bar{\theta}'\mu + \frac{\alpha^2}{2}\bar{\theta}'\Sigma\theta\right)\,.$$

Dividing by α_h, taking logs and rearranging yields

$$c_{h0} = -\frac{\log\delta_h}{\alpha_h} - \frac{\log R_f}{\alpha_h} + (w_{h0} - c_{h0} - p'\theta_h)R_f + \frac{\alpha}{\alpha_h}\bar{\theta}'\mu - \frac{\alpha^2}{2\alpha_h}\bar{\theta}'\Sigma\bar{\theta}\,.$$

By assumption, δ is the same for all h, so

$$\frac{\log\delta}{\alpha} = \sum_{h=1}^{H}\frac{\log\delta_h}{\alpha_h}\,.$$

Thus, aggregate demand for the consumption good can be expressed as

$$\sum_{h=1}^{H} c_{h0} = -\frac{\log\delta}{\alpha} - \frac{\log R_f}{\alpha} + R_f\sum_{h=1}^{H}(w_{h0} - c_{h0} - p'\theta_h) + \bar{\theta}'\mu - \frac{\alpha}{2}\bar{\theta}'\Sigma\bar{\theta}\,.$$

By market clearing for the risky assets, $\sum_{h=1}^H c_{h0} = \bar{c}$ if and only if $\sum_{h=1}^H w_{h0} - c_{h0} - p'\theta_h = 0$. Thus, each of these is equivalent to

$$\bar{c}_0 = -\frac{\log \delta}{\alpha} - \frac{\log R_f}{\alpha} + \bar{\theta}'\mu - \frac{\alpha}{2}\bar{\theta}'\Sigma\bar{\theta}\,.$$

The solution of this is

$$R_f = \frac{1}{\delta}\exp\left\{\alpha\left(\bar{\theta}'\mu - \bar{c}_0\right) - \frac{1}{2}\alpha^2\bar{\theta}'\Sigma\bar{\theta}\right\}\,.$$

(d) Explain in economic terms why the risk-free return (4.25) is higher when $\bar{\theta}'\mu$ is higher and lower when δ, $\bar{c}_0$, or $\bar{\theta}'\Sigma\bar{\theta}$ is higher.

Solution: $\bar{\theta}'\mu$ is expected aggregate date–1 consumption. Investors prefer to smooth consumption over time, so when $\bar{\theta}'\mu$ is larger, they wish to borrow to consume more at date 0. The risk-free return must rise to offset this inclination to borrow. The reverse is true when $\bar{c}_0$ is larger. When δ is higher, investors do not discount the future as much, and hence wish to save to finance date–1 consumption. The risk-free return must fall to offset this inclination to save. $\bar{\theta}'\Sigma\bar{\theta}$ is the variance of aggregate date–1 consumption. When it is larger, there is more risk, and investors expected date–1 utilities are smaller. They wish to transfer wealth from date 0 to date 1 in this circumstance, and the risk-free return must fall to offset that desire.

(e) Assume different investors may have different discount factors δ_h. Set $\tau_h = 1/\alpha_h$ and $\tau = \sum_{h=1}^H \tau_h$ (which is aggregate risk tolerance). Show that the equilibrium risk-free return is (4.25) when we define

$$\delta = \prod_{h=1}^H \delta_h^{\tau_h/\tau} \Leftrightarrow \log\delta = \sum_{h=1}^H \frac{\tau_h}{\tau}\log\delta_h$$

Solution: From the solution to Part (c), we have

$$c_{h0} = -\frac{\log \delta_h}{\alpha_h} - \frac{\log R_f}{\alpha_h} + (w_{h0} - c_{h0} - p'\theta_h)R_f + \frac{\alpha}{\alpha_h}\bar{\theta}'\mu - \frac{\alpha^2}{2\alpha_h}\bar{\theta}'\Sigma\bar{\theta}\,.$$

By the new definition of δ,

$$\log\delta = \sum_{h=1}^{H} \frac{\tau_h}{\tau} \log\delta_h \Rightarrow \frac{\log\delta}{\alpha} = \sum_{h=1}^{H} \frac{\log\delta_h}{\alpha_h}\,.$$

Thus, aggregate demand for the consumption good can be expressed as

$$\sum_{h=1}^{H} c_{h0} = -\frac{\log\delta}{\alpha} - \frac{\log R_f}{\alpha} + R_f \sum_{h=1}^{H}(w_{h0} - c_{h0} - p'\theta_h) + \bar{\theta}'\mu - \frac{\alpha}{2}\bar{\theta}'\Sigma\bar{\theta}\,.$$

This is the same as in Part (c), and the remainder of the proof is the same as that of Part (c).

4.2. Adopt the assumptions of the previous exercise, but assume there is no risk-free asset and there is consumption only at date 1. Show that the vector

$$p = \gamma\left(\mu - \alpha\Sigma\bar{\theta}\right)$$

is an equilibrium price vector for any $\gamma > 0$. Note: When $\gamma < 0$, this is also an equilibrium price vector, but each investor has a negative marginal value of wealth. In this model, investors are forced to hold assets because there is no date–0 consumption. When $\gamma < 0$, they are forced to invest in undesirable assets and would be better off if they had less wealth. Including consumption at date 0 or changing the budget constraint to $p'\theta \leq p'\bar{\theta}_h$ instead of $p'\theta = p'\bar{\theta}_h$ (i.e., allowing free disposal of wealth) eliminates the equilibria with $\gamma < 0$.

Solution: Investor h chooses θ_h to maximize the certainty equivalent

$$\theta'\mu - \frac{1}{2}\alpha\theta'\Sigma\theta$$

subject to the budget constraint $p'\theta = w_{h0}$. The optimum is

$$\theta_h = \frac{1}{\alpha_h}\Sigma^{-1}(\mu - \lambda_h p)$$

where λ_h is the Lagrange multiplier for the budget constraint. Thus,

$$\sum_{h=1}^{H} \theta_h = \frac{1}{\alpha}\Sigma^{-1}\mu - \left(\sum_{h=1}^{H} \frac{\lambda_h}{\alpha_h}\right)\Sigma^{-1}p\,.$$

To check if markets clear, we need to compute λ_h, which is determined by the budget equation of investor h:

$$w_{h0} = p'\theta_h = \frac{1}{\alpha_h} p'\Sigma^{-1}\mu - \frac{\lambda_h}{\alpha_h} p'\Sigma^{-1}p\,.$$

Instead of computing λ_h, it suffices to sum this over h, noting that aggregate initial wealth is $p'\bar{\theta}$. This yields

$$p'\bar{\theta} = \frac{1}{\alpha} p'\Sigma^{-1}\mu - \left(\sum_{h=1}^{H} \frac{\lambda_h}{\alpha_h}\right) p'\Sigma^{-1}p\,,$$

implying

$$\sum_{h=1}^{H} \frac{\lambda_h}{\alpha_h} = \frac{1}{\alpha}\left(\frac{p'\Sigma^{-1}\mu - \alpha p'\bar{\theta}}{p'\Sigma^{-1}p}\right)\,.$$

For $p = \gamma\left(\mu - \alpha\Sigma\bar{\theta}\right)$, the expression in parentheses in the last displayed equation is $1/\gamma$. Thus, for such prices, aggregate demand is

$$\frac{1}{\alpha}\Sigma^{-1}\mu - \frac{1}{\alpha\gamma}\Sigma^{-1}p = \frac{1}{\alpha}\Sigma^{-1}\mu - \frac{1}{\alpha\gamma}\Sigma^{-1}\left[\gamma\left(\mu - \alpha\Sigma\bar{\theta}\right)\right] = \bar{\theta}\,.$$

4.3. Suppose there are two investors, the first having constant relative risk aversion $\rho > 0$ and the second having constant relative risk aversion 2ρ.

(a) Show that the Pareto-optimal sharing rules are

$$\tilde{w}_1 = \tilde{w}_m + \eta - \sqrt{\eta^2 + 2\eta\tilde{w}_m}\,, \qquad \text{and} \qquad \tilde{w}_2 = \sqrt{\eta^2 + 2\eta\tilde{w}_m} - \eta\,,$$

for $\eta > 0$. Hint: Use the first-order condition and the quadratic formula. Because η is arbitrary in $(0,\infty)$, there are many equivalent ways to write the sharing rules.

Solution: The marginal utility of the first investor is $w^{-\rho}$, and the marginal utility of the second investor is $w^{-2\rho}$. The first-order condition is

$$\lambda_1\tilde{w}_1^{-\rho} = \lambda_2\tilde{w}_2^{-2\rho} = \lambda_2(\tilde{w}_m - \tilde{w}_1)^{-2\rho}\,.$$

This implies

$$\gamma \tilde{w}_1 = (\tilde{w}_m - \tilde{w}_1)^2\,,$$

where $\gamma = (\lambda_1/\lambda_2)^{-1/\rho}$. Thus

$$\tilde{w}_1^2 - (2\tilde{w}_m + \gamma)\tilde{w}_1 + \tilde{w}_m^2 = 0\,.$$

Applying the quadratic formula yields

$$\begin{aligned}\tilde{w}_1 &= \frac{2\tilde{w}_m + \gamma \pm \sqrt{(2\tilde{w}_m + \gamma)^2 - 4\tilde{w}_m^2}}{2} \\ &= \tilde{w}_m + \eta \pm \sqrt{\eta^2 + 2\eta\tilde{w}_m}\,,\end{aligned}$$

where $\eta = \gamma/2$. This implies

$$\tilde{w}_2 = \tilde{w}_m - \tilde{w}_1 = -\eta \pm \sqrt{\eta^2 + 2\eta\tilde{w}_m}\,.$$

To obtain $\tilde{w}_2 > 0$, we must have $\tilde{w}_2 = -\eta + \sqrt{\eta^2 + 2\eta\tilde{w}_m}$ and $\tilde{w}_1 = \tilde{w}_m + \eta - \sqrt{\eta^2 + 2\eta\tilde{w}_m}$.

(b) Suppose the market is complete and satisfies the law of one price. Show that the SDF in a competitive equilibrium is

$$\tilde{m} = \gamma\left(\sqrt{\eta^2 + 2\eta\tilde{w}_m} - \eta\right)^{-2\rho}$$

for positive constants γ and η.

Solution: Pareto optimality of the competitive equilibrium implies

$$\tilde{w}_2 = \sqrt{\eta^2 + 2\eta\tilde{w}_m} - \eta$$

for some $\eta > 0$. Thus, the second investor's marginal utility at a competitive equilibrium equals

$$\left(\sqrt{\eta^2 + 2\eta\tilde{w}_m} - \eta\right)^{-2\rho}\,.$$

The first-order condition for portfolio choice implies

$$\gamma\left(\sqrt{\eta^2 + 2\eta\tilde{w}_m} - \eta\right)^{-2\rho}$$

is an SDF for some $\gamma > 0$. By market completeness, this is the unique SDF.

4.4. Suppose each investor h has a concave utility function, and suppose an allocation $(\tilde{w}_1, \ldots, \tilde{w}_m)$ of market wealth $\tilde{w}_m$ satisfies the first-order condition

$$u_h'(\tilde{w}_h) = \gamma_h \tilde{m}$$

for each investor h, where $\tilde{m}$ is an SDF and is the same for each investor. Show that the allocation solves the social planner's problem (4.2) with weights $\lambda_h = 1/\gamma_h$.

Note: The first-order condition holds with the SDF being the same for each investor in a competitive equilibrium of a complete market, because there is a unique SDF in a complete market. Recall that γ_h in the first-order condition is the marginal value of beginning-of-period wealth (Section 3.1). Thus, the weights in the social planner's problem can be taken to be the reciprocals of the marginal values of wealth. Other things equal, investors with high wealth have low marginal values of wealth and hence have high weights in the social planner's problem.

Solution: By concavity and the first-order conditions, the allocation maximizes

$$\sum_{h=1}^{H} \frac{1}{\gamma_h} u_h(\tilde{w}_h) - \sum_{h=1}^{H} \tilde{m}\tilde{w}_h$$

in each state of the world, over all allocations $(\tilde{w}_1, \ldots, \tilde{w}_H)$, feasible or not. If $(\tilde{w}_1', \ldots, \tilde{w}_H')$ is any other feasible allocation, then

$$\sum_{h=1}^{H} \tilde{w}_h' = \tilde{w}_m = \sum_{h=1}^{H} \tilde{w}_h\,,$$

so

$$\sum_{h=1}^{H} \frac{1}{\gamma_h} u_h(\tilde{w}_h) - \sum_{h=1}^{H} \tilde{m}\tilde{w}_h \geq \sum_{h=1}^{H} \frac{1}{\gamma_h} u_h(\tilde{w}'_h) - \sum_{h=1}^{H} \tilde{m}\tilde{w}'_h$$
$$= \sum_{h=1}^{H} \frac{1}{\gamma_h} u_h(\tilde{w}'_h) - \sum_{h=1}^{H} \tilde{m}\tilde{w}_h \,.$$

Hence,

$$\sum_{h=1}^{H} \frac{1}{\gamma_h} u_h(\tilde{w}_h) \geq \sum_{h=1}^{H} \frac{1}{\gamma_h} u_h(\tilde{w}'_h) \,.$$

4.5. Suppose all investors have CARA utility. Consider an allocation

$$\tilde{w}_h = a_h + b_h \tilde{w}_m$$

where $b_h = \tau_h/\tau$ and $\sum_{h=1}^{H} a_h = 0$. Show that the allocation is Pareto optimal. Hint: Show that it solves the social planner's problem with weights λ_h defined as $\lambda_h = \tau_h \mathrm{e}^{a_h/\tau_h}$.

Solution: The first-order condition for the social planner's problem is

$$(\forall\, h) \quad \alpha_h \lambda_h \mathrm{e}^{-\alpha_h \tilde{w}_h} = \tilde{\eta} \,.$$

Setting $\tilde{w}_h = a_h + b_h \tilde{w}_m$ and $\lambda_h = \tau_h \mathrm{e}^{a_h/\tau_h}$, the left-hand side is $\mathrm{e}^{-\tau \tilde{w}_m}$. Thus, the first-order condition holds for $\tilde{\eta} = \mathrm{e}^{-\tau \tilde{w}_m}$. By concavity, the first-order condition is sufficient for optimality.

4.6. Suppose all investors have shifted CRRA utility with the same coefficient $\rho > 0$. Suppose $\tilde{w}_m > \zeta$. Consider an allocation

$$\tilde{w}_h = \zeta_h + b_h(\tilde{w}_m - \zeta)$$

where $\sum_{h=1}^{H} b_h = 1$. Show that the allocation is Pareto optimal. Hint: Show that it solves the social planner's problem with weights λ_h defined as $\lambda_h = b_h^{\rho}$.

Solution: The first-order condition for the social planner's problem is

$$(\forall\, h) \quad \lambda_h \tilde{w}_h^{-\rho} = \tilde{\eta} \,.$$