Chapter 3

Stochastic Discount Factors

3.1. Assume there are two possible states of the world: ω_1 and ω_2. There are two assets, a risk-free asset returning R_f in each state, and a risky asset with initial price equal to 1 and date–1 payoff $\tilde{x}$. Let $R_d = \tilde{x}(\omega_1)$ and $R_u = \tilde{x}(\omega_2)$. Assume without loss of generality that $R_u > R_d$.

(a) What inequalities between R_f, R_d and R_u are equivalent to the absence of arbitrage opportunities?

Solution: The payoff of a zero-cost portfolio is $\phi(\tilde{R} - R_f)$ for some ϕ. For this to be nonnegative in both states and positive in one state, we must have either (i) $\phi > 0$ and $R_u > R_d \geq R_f$ or (ii) $\phi < 0$ and $R_f \geq R_u > R_d$. Thus, a necessary and sufficient condition for the absence of arbitrage opportunities is that $R_u > R_f > R_d$.

(b) Assuming there are no arbitrage opportunities, compute the unique vector of state prices, and compute the unique risk-neutral probabilities of states ω_1 and ω_2.

Solution: Let q_d denote the state price of state ω_1 and q_u the state price of state ω_2. The state prices satisfy

$$q_d R_f + q_u R_f = 1\,,$$

$$q_d R_d + q_u R_u = 1\,.$$

The unique solution to this system of equations is

$$q_d = \frac{R_u - R_f}{R_f(R_u - R_d)}\,, \quad \text{and} \quad q_u = \frac{R_f - R_d}{R_f(R_u - R_d)}\,.$$

The risk neutral probabilities are $q_d R_f$ and $q_u R_f$.

(c) Suppose another asset is introduced into the market that pays $\max(\tilde{x} - K, 0)$ for some constant K. Compute the price at which this asset should trade, assuming there are no arbitrage opportunities.

Solution: The asset should trade at $q_u \max(x_u - K, 0) + q_d \max(x_d - K, 0)$, where x_d denotes the value of $\tilde{x}$ in state 1 and x_u the value of $\tilde{x}$ in state 2.

3.2. Assume there are three possible states of the world: ω_1, ω_2, and ω_3. Assume there are two assets: a risk-free asset returning R_f in each state, and a risky asset with return R_1 in state ω_1, R_2 in state ω_2, and R_3 in state ω_3. Assume the probabilities are 1/4 for state ω_1, 1/2 for state ω_2, and 1/4 for state ω_3. Assume $R_f = 1.0$, and $R_1 = 1.1$, $R_2 = 1.0$, and $R_3 = 0.9$.

(a) Prove that there are no arbitrage opportunities.

Solution: Let $\tilde{R}$ denote the risky asset return. A zero-cost portfolio has payoff $\phi(\tilde{R} - R_f)$ for some ϕ. This equals 0.1ϕ in state 1, 0 in state 2, and -0.1ϕ in state 3. Obviously, there is no ϕ such that $\phi(\tilde{R} - R_f)$ is nonnegative in all states and positive in some state.

(b) Describe the one-dimensional family of state-price vectors (q_1, q_2, q_3).

Solution: State prices must satisfy

$$q_1 + q_2 + q_3 = 1$$

$$1.1q_1 + q_2 + 0.9q_3 = 1\,.$$

Subtracting the top from the bottom shows that $q_3 = q_1$ and substituting this into the first shows that $q_2 = 1 - 2q_1$. q_1 is arbitrary.

(c) Describe the one-dimensional family of SDFs

$$\tilde{m} = (m_1, m_2, m_3)\,,$$

where m_i denotes the value of the SDF in state ω_i. Verify that $m_1 = 4$, $m_2 = -2$, $m_3 = 4$ is an SDF.

Solution: Stochastic discount factors are given by

$$m_1 = q_1/(1/4) = 4q_1\,, \quad m_2 = q_2/(1/2) = 2 - 4q_1\,, \quad m_3 = q_3/(1/4) = 4q_1\,,$$

with q_1 being arbitrary. Taking $q_1 = 1$ yields $m_1 = 4$, $m_2 = -2$, $m_3 = 4$.

(d) Consider the formula

$$\tilde{y}_p = \mathsf{E}[\tilde{y}] + \text{Cov}(\tilde{X}, \tilde{y})'\Sigma_x^{-1}(\tilde{X} - \mathsf{E}[\tilde{X}])$$

for the projection of a random variable $\tilde{y}$ onto the linear span of a constant and a random vector $\tilde{X}$. When the vector $\tilde{x}$ has only one component $\tilde{x}$ (is a scalar), the formula simplifies to

$$\tilde{y}_p = \mathsf{E}[\tilde{y}] + \beta(\tilde{x} - \mathsf{E}[\tilde{x}])\,,$$

where

$$\beta = \frac{\text{cov}(\tilde{x}, \tilde{y})}{\text{var}(\tilde{x})}\,.$$

Apply this formula with $\tilde{y}$ being the SDF $m_1 = 4$, $m_2 = -2$, $m_3 = 4$ and $\tilde{x}$ being the risky asset return $R_1 = 1.1$, $R_2 = 1.0$, $R_3 = 0.9$ to compute the projection of the SDF onto the span of the risk-free and risky assets.

Solution: We have $\mathsf{E}[\tilde{R}] = 1$ and $\mathsf{E}[\tilde{m}] = 1$ and

$$\text{cov}(\tilde{R}, \tilde{y}) = \frac{1}{4}(0.1)(3) + \frac{1}{2}(0)(-3) + \frac{1}{4}(-0.1)(3) = 0\,.$$

Thus, the projection is

$$\tilde{m}_p = \mathsf{E}[\tilde{m}] = 1\,.$$

(e) The projection in part (d) is by definition the payoff of some portfolio. What is the portfolio?

Solution: $\tilde{m}_p$ is the payoff of holding the risk-free asset.

3.3. Assume there is a risk-free asset. Let $\tilde{\mathbf{R}}$ denote the vector of risky asset returns, let μ denote the mean of $\tilde{\mathbf{R}}$, and let Σ denote the covariance matrix of $\tilde{\mathbf{R}}$. Let ι denote a vector of 1's. Derive the following formula for the SDF $\tilde{m}_p$ from the projection formula (3.32):

$$\tilde{m}_p = \frac{1}{R_f} + \left(\iota - \frac{1}{R_f}\mu\right)' \Sigma^{-1}(\tilde{\mathbf{R}} - \mu)\,.$$

Solution: From the projection formula we have:

$$\tilde{m}_p = \mathsf{E}[\tilde{m}_p] + \text{cov}(\tilde{m}_p, \tilde{\mathbf{R}})\Sigma^{-1}(\tilde{\mathbf{R}} - \mu)$$

When a risk-free asset exists, the mean of an SDF is $1/R_f$. Furthermore,

$$\text{cov}(\tilde{m}_p, \tilde{\mathbf{R}}) = \mathsf{E}[\tilde{m}_p(\tilde{\mathbf{R}} - \mu)'] = \mathsf{E}[\tilde{m}_p\tilde{\mathbf{R}}]' - \mathsf{E}[\tilde{m}_p]\mu' = \left(\iota - \frac{1}{R_f}\mu\right)'$$

Thus,

$$\tilde{m}_p = \frac{1}{R_f} + \left(\iota - \frac{1}{R_f}\mu\right)' \Sigma^{-1}(\tilde{\mathbf{R}} - \mu)\,.$$

3.4. Suppose two random vectors $\tilde{X}$ and $\tilde{Y}$ are joint normally distributed. Explain why the orthogonal projection (3.32) equals $\mathsf{E}[\tilde{Y}|\tilde{X}]$.

Solution: Let $\tilde{Y}_p$ denote the projection (3.32), so we have $\tilde{Y} = \tilde{Y}_p + \tilde{\varepsilon}$ with $\tilde{Y}_p$ being an affine function of $\tilde{X}$ and $\tilde{\varepsilon}$ being orthogonal to $\tilde{X}$. Then,

$$\mathsf{E}[\tilde{Y} \mid \tilde{X}] = \mathsf{E}[\tilde{Y}_p \mid \tilde{X}] + \mathsf{E}[\tilde{\varepsilon} \mid \tilde{X}] = \tilde{Y}_p + \mathsf{E}[\tilde{\varepsilon} \mid \tilde{X}]\,.$$

Now, because $\tilde{\varepsilon} = \tilde{Y} - \tilde{Y}_p$, which is a linear combination of the joint normal random vectors $\tilde{X}$ and $\tilde{Y}$, it follows that $\tilde{\varepsilon}$ and $\tilde{X}$ are joint normal. Hence, because they are uncorrelated, they are actually independent and consequently mean-independent. This implies that $\mathsf{E}[\tilde{\varepsilon} \mid \tilde{X}] = 0$, so

$$\mathsf{E}[\tilde{Y} \mid \tilde{X}] = \tilde{Y}_p\,.$$

3.5. Show that, if there is a strictly positive SDF, then there are no arbitrage opportunities.

Solution: Assume $\tilde{m}$ is a strictly positive SDF. If $\tilde{x}$ is a nonnegative marketed payoff, then its price is $\mathsf{E}[\tilde{m}\tilde{x}] \geq 0$, and $\mathsf{E}[\tilde{m}\tilde{x}] = 0$ if and only if $\tilde{x} = 0$ with probability one. Therefore, there are no arbitrage opportunities.

3.6. Show by example that the law of one price can hold but there can still be arbitrage opportunities.

Solution: Suppose there are two possible states of the world, and the market consists of the two Arrow securities having prices p_i. Then the market is complete, and each payoff $\tilde{x} = (x_1, x_2)$ has a unique cost $p_1x_1 + p_2x_2$. If $p_1 < 0$, then buying the first asset is an arbitrage opportunity.

3.7. Suppose there is an SDF $\tilde{m}$ with the property that for every function g there exists a portfolio θ (depending on g) such that

$$\sum_{i=1}^{n} \theta_i \tilde{x}_i = g(\tilde{m})\,.$$

Consider an investor with no labor income $\tilde{y}$. Show that his optimal wealth is a function of $\tilde{m}$. Hint: For any feasible $\tilde{w}$, define $\tilde{w}^* = \mathsf{E}[\tilde{w} \mid \tilde{m}]$, and show that $\tilde{w}^*$ is both budget feasible and at least as preferred as $\tilde{w}$, using the result of Section 1.5. Note: The assumption in this exercise is a weak form of market completeness. The exercise is inspired by Chamberlain (1988).

Solution: Set $\tilde{w}^* = \mathsf{E}[\tilde{w}|\tilde{m}]$ and $\tilde{\varepsilon} = \tilde{w} - \tilde{w}^*$, so we have that $\tilde{w}$ is $\tilde{w} = \tilde{w}^* + \tilde{\varepsilon}$. We will show that $\tilde{\varepsilon}$ has a zero mean and is mean-independent of $\tilde{w}^*$. Hence, the result of Section 1.5 shows that $\tilde{w}^*$ is at least as preferred as $\tilde{w}$. Finally, we will show that $\tilde{w}^*$ is budget feasible. This implies that $\tilde{w}^*$ is optimal. Since $\tilde{w}^* = \mathsf{E}[\tilde{w}|\tilde{m}]$, which is a function of $\tilde{m}$, this will complete the proof.

We have

$$\mathsf{E}[\tilde{\varepsilon} \mid \tilde{m}] = \mathsf{E}[\tilde{w} \mid \tilde{m}] - \mathsf{E}[\tilde{w}^* \mid \tilde{m}] = \tilde{w}^* - \tilde{w}^* = 0\,.$$

Also, because $\tilde{w}^*$ is a function of $\tilde{m}$,

$$\mathsf{E}[\tilde{\varepsilon} \mid \tilde{w}^*] = \mathsf{E}\left[\mathsf{E}[\tilde{\varepsilon} \mid \tilde{m}] \mid \tilde{w}^*\right] = 0\,.$$

Therefore, $\tilde{\varepsilon}$ has a zero mean and is mean-independent of $\tilde{w}^*$. Because $\tilde{w}^*$ is a function of $\tilde{m}$, there exists by assumption a portfolio $\tilde{\theta}$ with payoff equal to $\tilde{w}^*$. The cost of the portfolio is

$$\mathsf{E}[\tilde{m}\tilde{w}] = \mathsf{E}[\mathsf{E}[\tilde{m}\tilde{w} \mid \tilde{m}]] = \mathsf{E}[\tilde{m}\mathsf{E}[\tilde{w} \mid \tilde{m}]] = \mathsf{E}[\tilde{m}\tilde{w}^*]\,,$$

by iterated expectations. Hence, the cost of $\tilde{w}^*$ is the same as the cost of $\tilde{w}$, so $\tilde{w}^*$ is budget feasible.

3.8. Suppose there is a risk-free asset. Adopt the notation of Exercise 3.7, and assume the risky asset returns have a joint normal distribution. Show that the optimal portfolio of risky assets for an investor with no labor income is $\pi = \delta\Sigma^{-1}(\mu - R_f\iota)$ for some real number δ, by applying the reasoning of Exercise 3.7 with $\tilde{m} = \tilde{m}_p$, using the formula (3.45) for $\tilde{m}_p$ and using the results of Exercise 3.4.

Solution: For any budget feasible $\tilde{w}$, let $\tilde{w}^* = \mathsf{E}[\tilde{w} \mid \tilde{m}_p]$. Then, as shown in Exercise 3.7, $\tilde{w}$ equals $\tilde{w}^*$ plus mean-independent noise, so $\tilde{w}^*$ is preferred to $\tilde{w}$. Furthermore, $\tilde{w}^*$ is budget feasible. From (3.45),

$$\tilde{m}_p - \mathsf{E}[\tilde{m}_p] = -\frac{1}{R_f}(\mu - R_f\iota)'\Sigma^{-1}(\tilde{R}^{\text{vec}} - \mu)\,.$$

Hence,

$$\tilde{w}^* = \mathsf{E}[\tilde{w}] - \frac{1}{R_f}\left(\frac{\text{cov}(\tilde{w}, \tilde{m}_p)}{\text{var}(\tilde{m}_p)}\right)(\mu - R_f\iota)'\Sigma^{-1}(\tilde{R}^{\text{vec}} - \mu)\,.$$

This shows that the portfolio of risky assets producing $\tilde{w}^*$ is $\delta\Sigma^{-1}(\mu - R_f\iota)$ for

$$\delta = -\frac{1}{R_f}\left(\frac{\text{cov}(\tilde{w}, \tilde{m}_p)}{\text{var}(\tilde{m}_p)}\right) = -\frac{1}{R_f}\left(\frac{\text{cov}(\tilde{w}^*, \tilde{m}_p)}{\text{var}(\tilde{m}_p)}\right)\,,$$

the second equality following from iterated expectations.

3.9. Assume there is a finite number of assets, and the payoff of each asset has a finite variance. Assume the Law of One Price holds. Apply facts stated in Section 3.8 to show that there is a unique SDF $\tilde{m}_p$ in the span of the asset payoffs. Show that the orthogonal projection of any other SDF onto the span of the asset payoffs equals $\tilde{m}_p$.

Solution: The span of the assets is a finite-dimensional subspace of $\mathcal{L}^2$. The law of one price states that there is a unique price $C[\tilde{x}]$ for each $\tilde{x}$ in the span of the payoffs. The function $C[\cdot]$ is linear. Therefore, it has a Riesz representation $C[\tilde{x}] = \mathsf{E}[\tilde{x}\tilde{m}_p]$ for a unique $\tilde{m}_p$ in the span of the assets. Given any stochastic discount factor $\tilde{m}$, we have $\tilde{m} = \tilde{m}^* + \tilde{\varepsilon}$, where the orthogonal projection $\tilde{m}^*$ is in the span of the assets and $\tilde{\varepsilon}$ is orthogonal to the span of the assets. Hence, $C[\tilde{x}] = \mathsf{E}[\tilde{m}\tilde{x}] = \mathsf{E}[\tilde{x}\tilde{m}^*]$ for all $\tilde{x}$ in the span of the assets. Thus, $\tilde{m}^*$ is also in the span of the assets and represents the price function. By the uniqueness of the Riesz representation, it must be that $\tilde{m}^* = \tilde{m}_p$.