2.7. Consider the portfolio choice problem with only a risk-free asset and with consumption at both the beginning and end of the period. Assume the investor has time-additive power utility, so he solves

$$\max \quad \frac{1}{1-\rho}c_0^{1-\rho} + \delta\frac{1}{1-\rho}c_1^{1-\rho} \quad \text{subject to} \quad c_0 + \frac{1}{R_f}c_1 = w_0\,.$$

As shown in Exercise 2.6, the investor's EIS is $1/\rho$.

(a) Show that the optimal consumption-to-wealth ratio c_0/w_0 is a decreasing function of R_f if the EIS is greater than 1 and an increasing function of R_f if the EIS is less than 1. Note: the effect of changing R_f is commonly broken into an income effect and substitution effect. This shows that the substitution effect dominates when the EIS is high and the income effect dominates when the EIS is low.

Solution: Substituting the budget constraint, the objective function is

$$\frac{1}{1-\rho}c_0^{1-\rho} + \delta\frac{1}{1-\rho}R_f^{1-\rho}(w_0-c_0)^{1-\rho}\,,$$

and the first-order condition is

$$c_0^{-\rho} - \delta R_f^{1-\rho}(w_0-c_0)^{-\rho} = 0\,.$$

This implies

$$c_0 = \delta^{-1/\rho}R_f^{1-1/\rho}(w_0-c_0)\,,$$

so

$$c_0 = \frac{\delta^{-1/\rho}R_f^{1-1/\rho}}{1+\delta^{-1/\rho}R_f^{1-1/\rho}}w_0\,.$$

The factor

$$\frac{\delta^{-1/\rho}R_f^{1-1/\rho}}{1+\delta^{-1/\rho}R_f^{1-1/\rho}}$$

is an increasing function of R_f if $1-1/\rho>0$ and a decreasing function of R_f if $1-1/\rho<0$.

(b) For given c_0 and $\tilde{c}_1$, show that the solution of the investor's optimization problem implies that R_f must be lower when the EIS is higher. **This exercise needs the additional assumption that $c_1 > c_0$. Also, there shouldn't be a tilde on c_1, because it is not random.**

Solution: From the solution to Part (a), we have

$$c_0 = \frac{\delta^{-1/\rho} R_f^{1-1/\rho}}{1 + \delta^{-1/\rho} R_f^{1-1/\rho}} w_0 \,.$$

Using this and the budget constraint, we obtain

$$c_1 = \frac{R_f}{1 + \delta^{-1/\rho} R_f^{1-1/\rho}} w_0 \,.$$

Thus,

$$\frac{c_1}{c_0} = \delta^{1/\rho} R_f^{1/\rho} \,.$$

This implies

$$R_f = \frac{1}{\delta} \left(\frac{c_1}{c_0} \right)^{\rho} \,.$$

Thus, under the assumption $c_1 > c_0$, R_f is an increasing function of ρ and hence a decreasing function of the EIS.

2.8. Consider the portfolio choice problem with only a risk-free asset and with consumption at both the beginning and end of the period. Suppose the investor has time-additive utility with $u_0 = u$ and $u_1 = \delta u$ for a common function u and discount factor δ. Suppose the investor has a random endowment $\tilde{y}$ at the end of the period, so he chooses c_0 to maximize

$$u(c_0) + \delta \mathsf{E}[u((w_0 - c_0)R_f + \tilde{y})] \,.$$

Suppose the investor has convex marginal utility ($u''' > 0$) and suppose that $\mathsf{E}[\tilde{y}] = 0$. Show that the optimal c_0 is smaller than if $\tilde{y} = 0$. Note: This illustrates the concept of precautionary savings—the risk imposed by $\tilde{y}$ results in higher savings $w_0 - c_0$.

Solution: The first-order condition is that

$$u'(c_0) = \delta \mathsf{E}[u'((w_0 - c_0)R_f + \tilde{y})]\,.$$

By Jensen's inequality and the convexity of u',

$$\mathsf{E}[u'((w_0 - c_0)R_f + \tilde{y})] > u'\left(\mathsf{E}[(w_0 - c_0)R_f + \tilde{y}]\right) = u'((w_0 - c_0)R_f)\,.$$

Thus,

$$u'(c_0) > \delta u'((w_0 - c_0)R_f)\,.$$

The first-order condition if $\tilde{y} = 0$ is for these to be equal. Because the left-hand side is decreasing in c_0 and the right-hand side increasing in c_0, equality requires that c_0 be increased. Thus, the optimal c_0 would be larger if $\tilde{y} = 0$.

2.9. Letting c_0^* denote optimal consumption in the previous problem, define the precautionary premium π by

$$u'((w_0 - \pi - c_0^*)R_f) = \mathsf{E}[u'((w_0 - c_0^*)R_f + \tilde{y})]\,.$$

(a) Show that c_0^* would be the optimal consumption of the investor if he had no end-of-period endowment and had initial wealth $w_0 - \pi$.

Solution: The first-order condition is that

$$u'(c_0^*) = \delta \mathsf{E}[u'((w_0 - c_0^*)R_f + \tilde{y})]\,.$$

By the definition of the precautionary premium, this implies

$$u'(c_0^*) = \delta u'((w_0 - \pi - c_0^*)R_f)\,.$$

This is the first-order condition for initial wealth $w_0 - \pi$ when $\tilde{y} = 0$.

(b) Assume the investor has CARA utility. Show that the precautionary premium is independent of initial wealth (again, no wealth effects with CARA utility).

Solution: With CARA utility $-\mathrm{e}^{-\alpha w}$, the marginal utility is $\alpha \mathrm{e}^{-\alpha w}$. Therefore the precautionary premium is π satisfying

$$\alpha \mathrm{e}^{-\alpha(w_0-\pi-c_0^*)R_f} = \alpha \mathsf{E}\left[\mathrm{e}^{-\alpha\left((w_0-c_0^*)R_f+\tilde{y}\right)}\right] .$$

Multiplying by $\mathrm{e}^{\alpha(w_0-c_0^*)R_f}/\alpha$ yields

$$\mathrm{e}^{\alpha\pi R_f} = \mathsf{E}\left[\mathrm{e}^{-\alpha\tilde{y}}\right] ,$$

with solution

$$\pi = \frac{1}{\alpha R_f} \log \mathsf{E}\left[\mathrm{e}^{-\alpha\tilde{y}}\right] .$$

Chapter 3

Stochastic Discount Factors

3.1. Assume there are two possible states of the world: ω_1 and ω_2. There are two assets, a risk-free asset returning R_f in each state, and a risky asset with initial price equal to 1 and date–1 payoff $\tilde{x}$. Let $R_d = \tilde{x}(\omega_1)$ and $R_u = \tilde{x}(\omega_2)$. Assume without loss of generality that $R_u > R_d$.

(a) What inequalities between R_f, R_d and R_u are equivalent to the absence of arbitrage opportunities?

Solution: The payoff of a zero-cost portfolio is $\phi(\tilde{R} - R_f)$ for some ϕ. For this to be nonnegative in both states and positive in one state, we must have either (i) $\phi > 0$ and $R_u > R_d \geq R_f$ or (ii) $\phi < 0$ and $R_f \geq R_u > R_d$. Thus, a necessary and sufficient condition for the absence of arbitrage opportunities is that $R_u > R_f > R_d$.

(b) Assuming there are no arbitrage opportunities, compute the unique vector of state prices, and compute the unique risk-neutral probabilities of states ω_1 and ω_2.

Solution: Let q_d denote the state price of state ω_1 and q_u the state price of state ω_2. The state prices satisfy

$$q_d R_f + q_u R_f = 1\,,$$

$$q_d R_d + q_u R_u = 1\,.$$

The unique solution to this system of equations is

$$q_d = \frac{R_u - R_f}{R_f(R_u - R_d)}\,, \quad \text{and} \quad q_u = \frac{R_f - R_d}{R_f(R_u - R_d)}\,.$$

The risk neutral probabilities are $q_d R_f$ and $q_u R_f$.

(c) Suppose another asset is introduced into the market that pays $\max(\tilde{x} - K, 0)$ for some constant K. Compute the price at which this asset should trade, assuming there are no arbitrage opportunities.

Solution: The asset should trade at $q_u \max(x_u - K, 0) + q_d \max(x_d - K, 0)$, where x_d denotes the value of $\tilde{x}$ in state 1 and x_u the value of $\tilde{x}$ in state 2.

3.2. Assume there are three possible states of the world: ω_1, ω_2, and ω_3. Assume there are two assets: a risk-free asset returning R_f in each state, and a risky asset with return R_1 in state ω_1, R_2 in state ω_2, and R_3 in state ω_3. Assume the probabilities are $1/4$ for state ω_1, $1/2$ for state ω_2, and $1/4$ for state ω_3. Assume $R_f = 1.0$, and $R_1 = 1.1$, $R_2 = 1.0$, and $R_3 = 0.9$.

(a) Prove that there are no arbitrage opportunities.

Solution: Let $\tilde{R}$ denote the risky asset return. A zero-cost portfolio has payoff $\phi(\tilde{R} - R_f)$ for some ϕ. This equals 0.1ϕ in state 1, 0 in state 2, and -0.1ϕ in state 3. Obviously, there is no ϕ such that $\phi(\tilde{R} - R_f)$ is nonnegative in all states and positive in some state.

(b) Describe the one-dimensional family of state-price vectors (q_1, q_2, q_3).

Solution: State prices must satisfy

$$q_1 + q_2 + q_3 = 1$$

$$1.1q_1 + q_2 + 0.9q_3 = 1\,.$$

Subtracting the top from the bottom shows that $q_3 = q_1$ and substituting this into the first shows that $q_2 = 1 - 2q_1$. q_1 is arbitrary.

(c) Describe the one-dimensional family of SDFs

$$\tilde{m} = (m_1, m_2, m_3)\,,$$

where m_i denotes the value of the SDF in state ω_i. Verify that $m_1 = 4$, $m_2 = -2$, $m_3 = 4$ is an SDF.

Solution: Stochastic discount factors are given by

$$m_1 = q_1/(1/4) = 4q_1\,, \quad m_2 = q_2/(1/2) = 2 - 4q_1\,, \quad m_3 = q_3/(1/4) = 4q_1\,,$$

with q_1 being arbitrary. Taking $q_1 = 1$ yields $m_1 = 4$, $m_2 = -2$, $m_3 = 4$.

(d) Consider the formula

$$\tilde{y}_p = \mathsf{E}[\tilde{y}] + \text{Cov}(\tilde{X}, \tilde{y})'\Sigma_x^{-1}(\tilde{X} - \mathsf{E}[\tilde{X}])$$

for the projection of a random variable $\tilde{y}$ onto the linear span of a constant and a random vector $\tilde{X}$. When the vector $\tilde{x}$ has only one component $\tilde{x}$ (is a scalar), the formula simplifies to

$$\tilde{y}_p = \mathsf{E}[\tilde{y}] + \beta(\tilde{x} - \mathsf{E}[\tilde{x}])\,,$$

where

$$\beta = \frac{\text{cov}(\tilde{x}, \tilde{y})}{\text{var}(\tilde{x})}\,.$$