Chapter 2

Portfolio Choice

2.1. Suppose there is a risk-free asset with $R_f = 1.05$ and three risky assets each of which has an expected return equal to 1.10. Suppose the covariance matrix of the risky asset returns is

$$\Sigma = \begin{pmatrix} 0.09 & 0.06 & 0 \\ 0.06 & 0.09 & 0 \\ 0 & 0 & 0.09 \end{pmatrix} .$$

Suppose the returns are normally distributed. What is the optimal fraction of wealth to invest in each of the risky assets for a CARA investor with $\alpha w_0 = 2$? Why is the optimal investment higher for the third asset than for the other two?

Solution: The optimal portfolio of a CARA investor with multiple risky assets can be calculated by the formula (2.22). The optimal fraction of initial wealth is:

$$\begin{aligned}\pi &= \frac{1}{\alpha w_0}\Sigma^{-1}(\mu - R_f\iota) \\ &= \frac{1}{2}\begin{pmatrix}0.09 & 0.06 & 0\\ 0.06 & 0.09 & 0\\ 0 & 0 & 0.09\end{pmatrix}^{-1}\left[\begin{pmatrix}1.10\\1.10\\1.10\end{pmatrix} - \begin{pmatrix}1.05\\1.05\\1.05\end{pmatrix}\right] = \begin{pmatrix}3.5\\3.5\\5.83\end{pmatrix}\end{aligned}$$

The risky assets have the same variance and the same expected return, but the first two assets are positively correlated. Lower risk is achieved by holding more of the third asset, which is uncorrelated with the others.

2.2. Suppose there is a risk-free asset and n risky assets with payoffs $\tilde{x}_i$ and prices p_i. Assume the vector $\tilde{x} = (\tilde{x}_1 \cdots \tilde{x}_n)'$ is normally distributed with mean μ_x and nonsingular covariance matrix Σ_x. Let $p = (p_1 \cdots p_n)'$. Suppose there is consumption at date 0 and consider an investor with initial wealth w_0 and CARA utility at date 1:

$$u_1(c) = -\mathrm{e}^{-\alpha c}\,.$$

Let θ_i denote the number of shares the investor considers holding of asset i and set $\theta = (\theta_1 \cdots \theta_n)'$. The investor chooses consumption c_0 at date 0 and a portfolio θ, producing wealth $(w_0 - c_0 - \theta' p)R_f + \theta'\tilde{x}$ at date 1.

(a) Show that the optimal vector of share holdings is

$$\theta = \frac{1}{\alpha}\Sigma_x^{-1}(\mu_x - R_f p)\,.$$

Solution: The investor chooses date–0 consumption c_0 and a portfolio θ of risky assets to maximize

$$u(c_0) - \exp\left(-\alpha(w_0 - c_0 - \theta' p)R_f - \alpha\theta'\mu_x + \frac{1}{2}\alpha^2\theta'\Sigma_x\theta\right)\,.$$

The optimal portfolio θ is the portfolio that maximizes

$$-R_f p'\theta + \mu_x'\theta - \frac{1}{2}\alpha\theta'\Sigma_x\theta \,.$$

The first-order condition is

$$-R_f p + \mu_x - \alpha\Sigma_x\theta = 0 \,,$$

with solution

$$\theta = \frac{1}{\alpha}\Sigma_x^{-1}(\mu_x - R_f p) \,.$$

(b) Suppose all of the asset prices are positive, so we can define returns $\tilde{x}_i/p_i$. Explain why (2.33) implies (2.22). Note: This is another illustration of the absence of wealth effects. Neither date-0 wealth nor date-0 consumption affects the optimal portfolio for a CARA investor.

Solution: Let P denote the $n \times n$ diagonal matrix with the ith diagonal element being p_i. Then $\phi = P\theta$, $\mu - R_f\iota = P^{-1}(\mu_x - R_f p)$, and $\Sigma = P^{-1}\Sigma_x P^{-1}$. Therefore, multiplying (2.33) by P gives

$$\phi = \frac{1}{\alpha}P(P\Sigma P)^{-1}P(\mu - R_f\iota) = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f\iota) \,.$$

2.3. Suppose there is a risk-free asset with return R_f and a risky asset with return $\tilde{R}$. Consider an investor who maximizes expected end-of-period utility of wealth and who has CARA utility and invests w_0. Suppose the investor has a random endowment $\tilde{y}$ at the end of the period, so his end-of-period wealth is $\phi_f R_f + \phi\tilde{R} + \tilde{y}$, where ϕ_f denotes the investment in the risk-free asset and ϕ the investment in the risky asset.

(a) Suppose $\tilde{y}$ and $\tilde{R}$ are independent. Show that the optimal ϕ is the same as if there were no end-of-period endowment. Hint: Use the law of iterated expectations as in Section 1.5 and the fact that if $\tilde{v}$ and $\tilde{x}$ are independent random variables then $\mathsf{E}[\tilde{v}\tilde{x}] = \mathsf{E}[\tilde{v}]\mathsf{E}[\tilde{x}]$.

Solution: The expected utility is

$$-\mathsf{E}\left[\exp\left(-\alpha w_0 R_f - \alpha\phi\left(\tilde{R} - R_f\right) - \alpha\tilde{y}\right)\right] = -\mathsf{E}\left[\mathrm{e}^{-\alpha w_0 R_f - \alpha\phi\left(\tilde{R}-R_f\right)}\mathrm{e}^{-\alpha\tilde{y}}\right] .$$

By independence, this equals

$$-\mathsf{E}\left[\mathrm{e}^{-\alpha w_0 R_f - \alpha\phi\left(\tilde{R}-R_f\right)}\right]\mathsf{E}\left[\mathrm{e}^{-\alpha\tilde{y}}\right] ,$$

and maximizing this over ϕ is equivalent to maximizing

$$-\mathsf{E}\left[\mathrm{e}^{-\alpha w_0 R_f - \alpha\phi\left(\tilde{R}-R_f\right)}\right] ,$$

which is the same as if $\tilde{y} = 0$.

(b) Define $b = \operatorname{cov}(\tilde{y}, \tilde{R})/\operatorname{var}(\tilde{R})$, $a = (\mathsf{E}[\tilde{y}] - b\mathsf{E}[\tilde{R}])/R_f$ and $\tilde{\varepsilon} = \tilde{y} - aR_f - b\tilde{R}$. Show that $\tilde{y} = aR_f + b\tilde{R} + \tilde{\varepsilon}$ and that $\tilde{\varepsilon}$ has a zero mean and is uncorrelated with $\tilde{R}$. Note: This is an example of an orthogonal projection, which is discussed in more generality in Section 3.5.

Solution: From the definition of $\tilde{\varepsilon}$, we have $\tilde{y} = aR_f + b\tilde{R} + \tilde{\varepsilon}$. We need to show that $\tilde{\varepsilon}$ has a zero mean and is uncorrelated with $\tilde{R}$. We have

$$\mathsf{E}[\varepsilon] = \mathsf{E}[\tilde{y} - aR_f - b\tilde{R}] = \mathsf{E}[\tilde{y}] - [(\mathsf{E}[\tilde{y}] - b\mathsf{E}[\tilde{R}])/R_f].R_f - b\mathsf{E}[\tilde{R}] = 0\,.$$

Furthermore,

$$\begin{aligned}\operatorname{cov}(\tilde{\varepsilon}, \tilde{R}) &= \operatorname{cov}(\tilde{y} - aR_f - b\tilde{R}, \tilde{R}) \\ &= \operatorname{cov}(\tilde{y}, \tilde{R}) - b\operatorname{var}(\tilde{R}) \\ &= 0\,,\end{aligned}$$

using the definition of b for the last equality.

(c) Suppose $\tilde{y}$ and $\tilde{R}$ have a joint normal distribution. Using the result of the previous part, show that the optimal ϕ is $\phi^* - b$, where ϕ^* denotes the optimal investment in the risky asset when there is no end-of-period endowment.

Solution: The expected end-of-period wealth is

$$w_0 R_f + \phi(\mathsf{E}[\tilde{R}] - R_f + \mathsf{E}[\tilde{y}] = (w_0 + a - \phi)R_f + (\phi + b)\mathsf{E}[\tilde{R}]\,,$$

and the variance of end-of-period wealth is

$$\phi^2 \operatorname{var}(\tilde{R}) + 2\phi \operatorname{cov}(\tilde{R}, \tilde{y}) + \operatorname{var}(\tilde{y}) = (\phi^2 + 2\phi b + b^2)\operatorname{var}(\tilde{R}) + \operatorname{var}(\tilde{\varepsilon})\,.$$

The expected utility is

$$-\exp\left(-\alpha\left((w_0 + a - \phi)R_f + (\phi + b)\mathsf{E}[\tilde{R}]\right) + \frac{1}{2}\alpha^2\left((\phi^2 + 2\phi b + b^2)\operatorname{var}(\tilde{R}) + \operatorname{var}(\tilde{\varepsilon})\right)\right)\,.$$

Maximizing this over ϕ is equivalent to maximizing

$$\phi(\mathsf{E}[\tilde{R}] - R_f) - \frac{1}{2}\alpha(\phi^2 + 2\phi b)\operatorname{var}(\tilde{R})\,,$$

for which the solution is

$$\phi = \frac{\mathsf{E}[\tilde{R}] - R_f}{\alpha \operatorname{var}(\tilde{R})} - b\,.$$

2.4. Consider a CARA investor with n risky assets having normally distributed returns, as studied in Section 2.4, but suppose there is no risk-free asset, so the budget constraint is $\iota'\phi = w_0$. Show that the optimal portfolio is

$$\phi = \frac{1}{\alpha}\Sigma^{-1}\mu + \left(\frac{\alpha w_0 - \iota'\Sigma^{-1}\mu}{\alpha\iota'\Sigma^{-1}\iota}\right)\Sigma^{-1}\iota\,.$$

Note: As will be seen in Section 5.2, the two vectors $\Sigma^{-1}\mu$ and $\Sigma^{-1}\iota$ play an important role in mean-variance analysis even without the CARA/normal assumption.

Solution: The expected payoff of a portfolio ϕ is $\phi'\mu$ and the variance is $\phi'\Sigma\phi$. The expected utility is

$$-\exp\left(-\alpha\phi'\mu + \frac{1}{2}\alpha^2\phi'\Sigma\phi\right)\,.$$

Maximizing this is equivalent to maximizing

$$\phi'\mu - \frac{1}{2}\alpha\phi'\Sigma\phi\,.$$

Let λ denote the Lagrange multiplier for the constraint $\iota'\phi = w_0$. The Lagrangean is

$$\phi'(\mu - \lambda\iota) - \frac{1}{2}\alpha\phi'\Sigma\phi\,,$$

and the first-order condition is

$$\mu - \lambda\iota - \alpha\Sigma\phi = 0\,,$$

which is solved by

$$\phi = \frac{1}{\alpha}\Sigma^{-1}(\mu - \lambda\iota)\,.$$

Imposing the constraint $\iota'\phi = w_0$ yields

$$\frac{1}{\alpha}\iota'\Sigma^{-1}\mu - \frac{\lambda}{\alpha}\iota'\Sigma^{-1}\iota = w_0\,.$$

Therefore,

$$\lambda = \frac{\iota'\Sigma^{-1}\mu - \alpha w_0}{\iota'\Sigma^{-1}\iota}\,,$$

and

$$\phi = \frac{1}{\alpha}\Sigma^{-1}\mu + \left(\frac{\alpha w_0 - \iota'\Sigma^{-1}\mu}{\alpha\iota'\Sigma^{-1}\iota}\right)\Sigma^{-1}\iota\,.$$

2.5. Suppose there is a risk-free asset and n risky assets. Consider an investor with quadratic utility who seeks to maximize

$$\zeta\mathsf{E}[\tilde{w}] - \frac{1}{2}\mathsf{E}[\tilde{w}]^2 - \frac{1}{2}\,\mathrm{var}(\tilde{w})\,.$$

Show that the optimal portfolio for the investor is

$$\phi = \frac{1}{1+\kappa^2}(\zeta - w_0R_f)\Sigma^{-1}(\mu - R_f\iota)\,,$$

where

$$\kappa^2 = (\mu - R_f\iota)'\Sigma^{-1}(\mu - R_f\iota)\,.$$

Hint: In the first-order conditions, define $\gamma = (\mu - R_f\iota)'\phi$, solve for ϕ in terms of γ, and then compute γ. Note: We will see in Chapter 5 that κ is the maximum Sharpe ratio of any portfolio.

Solution: The expected payoff of a portfolio ϕ of risky assets is $w_0R_f + \phi'(\mu - R_f\iota)$, and the variance is $\phi'\Sigma\phi$. The expected utility is

$$\begin{aligned}
&\zeta[w_0R_f + \phi'(\mu - R_f\iota)] - \frac{1}{2}[w_0R_f + \phi'(\mu - R_f\iota)]^2 - \frac{1}{2}\phi'\Sigma\phi \\
&\quad = \zeta[w_0R_f + \phi'(\mu - R_f\iota)] - \frac{1}{2}w_0^2R_f^2 - w_0R_f\phi'(\mu - R_f\iota) - \frac{1}{2}\phi'(\mu - R_f\iota)(\mu - R_f\iota)'\phi - \frac{1}{2}\phi'\Sigma\phi\,.
\end{aligned}$$

The first-order condition for maximizing this is

$$\zeta(\mu - R_f\iota) - w_0R_f(\mu - R_f\iota) - (\mu - R_f\iota)(\mu - R_f\iota)'\phi - \Sigma\phi = 0\,.$$

Setting $\gamma = (\mu - R_f\iota)'\phi$, we have

$$\zeta(\mu - R_f\iota) - w_0R_f(\mu - R_f\iota) - \gamma(\mu - R_f\iota) - \Sigma\phi = 0\,,$$

with solution

$$\phi = (\zeta - w_0R_f - \gamma)\Sigma^{-1}(\mu - R_f\iota)\,.$$

Thus,

$$\begin{aligned}
\gamma &= (\zeta - w_0R_f - \gamma)(\mu - R_f\iota)'\Sigma^{-1}(\mu - R_f\iota) \\
&= (\zeta - w_0R_f - \gamma)\kappa^2\,,
\end{aligned}$$

implying

$$\gamma = \frac{\kappa^2}{1+\kappa^2}(\zeta - w_0R_f)\,,$$

and

$$\phi = \frac{1}{1+\kappa^2}(\zeta - w_0 R_f)\Sigma^{-1}(\mu - R_f \iota)\,.$$

2.6. Consider a utility function $v(c_0, c_1)$. The marginal rate of substitution (MRS) is defined to be the negative of the slope of an indifference curve and is equal to

$$\text{MRS}(c_0, c_1) = \frac{\partial v(c_0, c_1)/\partial c_0}{\partial v(c_0, c_1)/\partial c_1}\,.$$

The elasticity of intertemporal substitution is defined as

$$\frac{\mathrm{d}\log(c_1/c_0)}{\mathrm{d}\log \text{MRS}(c_0, c_1)}\,,$$

where the marginal rate of substitution is varied holding utility constant. Show that, if

$$v(c_0, c_1) = \frac{1}{1-\rho}c_0^{1-\rho} + \frac{\delta}{1-\rho}c_1^{1-\rho}\,,$$

then the EIS is $1/\rho$.

Solution: Holding utility constant implies

$$c_0^{-\rho}\,\mathrm{d}c_0 + \delta c_1^{-\rho}\,\mathrm{d}c_1 = 0\,,$$

so

$$-\frac{\mathrm{d}c_1}{\mathrm{d}c_0} = \frac{1}{\delta}\left(\frac{c_0}{c_1}\right)^{-\rho}\,.$$

This is the marginal rate of substitution. Setting $x = c_1/c_0$, we have

$$\log \text{MRS} = -\log\delta + \rho\log x\,.$$

Hence,

$$\frac{\mathrm{d}\log \text{MRS}}{\mathrm{d}\log x} = \rho\,.$$

The elasticity of intertemporal substitution is the reciprocal $1/\rho$.

2.7. Consider the portfolio choice problem with only a risk-free asset and with consumption at both the beginning and end of the period. Assume the investor has time-additive power utility, so he solves

$$\max \quad \frac{1}{1-\rho}c_0^{1-\rho} + \delta\frac{1}{1-\rho}c_1^{1-\rho} \quad \text{subject to} \quad c_0 + \frac{1}{R_f}c_1 = w_0\,.$$

As shown in Exercise 2.6, the investor's EIS is $1/\rho$.

(a) Show that the optimal consumption-to-wealth ratio c_0/w_0 is a decreasing function of R_f if the EIS is greater than 1 and an increasing function of R_f if the EIS is less than 1. Note: the effect of changing R_f is commonly broken into an income effect and substitution effect. This shows that the substitution effect dominates when the EIS is high and the income effect dominates when the EIS is low.

Solution: Substituting the budget constraint, the objective function is

$$\frac{1}{1-\rho}c_0^{1-\rho} + \delta\frac{1}{1-\rho}R_f^{1-\rho}(w_0-c_0)^{1-\rho}\,,$$

and the first-order condition is

$$c_0^{-\rho} - \delta R_f^{1-\rho}(w_0-c_0)^{-\rho} = 0\,.$$

This implies

$$c_0 = \delta^{-1/\rho}R_f^{1-1/\rho}(w_0-c_0)\,,$$

so

$$c_0 = \frac{\delta^{-1/\rho}R_f^{1-1/\rho}}{1+\delta^{-1/\rho}R_f^{1-1/\rho}}w_0\,.$$

The factor

$$\frac{\delta^{-1/\rho}R_f^{1-1/\rho}}{1+\delta^{-1/\rho}R_f^{1-1/\rho}}$$

is an increasing function of R_f if $1-1/\rho>0$ and a decreasing function of R_f if $1-1/\rho<0$.

(b) For given c_0 and $\tilde{c}_1$, show that the solution of the investor's optimization problem implies that R_f must be lower when the EIS is higher. **This exercise needs the additional assumption that $c_1 > c_0$. Also, there shouldn't be a tilde on c_1, because it is not random.**

Solution: From the solution to Part (a), we have

$$c_0 = \frac{\delta^{-1/\rho} R_f^{1-1/\rho}}{1 + \delta^{-1/\rho} R_f^{1-1/\rho}} w_0 \,.$$

Using this and the budget constraint, we obtain

$$c_1 = \frac{R_f}{1 + \delta^{-1/\rho} R_f^{1-1/\rho}} w_0 \,.$$

Thus,

$$\frac{c_1}{c_0} = \delta^{1/\rho} R_f^{1/\rho} \,.$$

This implies

$$R_f = \frac{1}{\delta} \left(\frac{c_1}{c_0} \right)^{\rho} \,.$$

Thus, under the assumption $c_1 > c_0$, R_f is an increasing function of ρ and hence a decreasing function of the EIS.

2.8. Consider the portfolio choice problem with only a risk-free asset and with consumption at both the beginning and end of the period. Suppose the investor has time-additive utility with $u_0 = u$ and $u_1 = \delta u$ for a common function u and discount factor δ. Suppose the investor has a random endowment $\tilde{y}$ at the end of the period, so he chooses c_0 to maximize

$$u(c_0) + \delta \mathsf{E}[u((w_0 - c_0)R_f + \tilde{y})] \,.$$

Suppose the investor has convex marginal utility ($u''' > 0$) and suppose that $\mathsf{E}[\tilde{y}] = 0$. Show that the optimal c_0 is smaller than if $\tilde{y} = 0$. Note: This illustrates the concept of precautionary savings—the risk imposed by $\tilde{y}$ results in higher savings $w_0 - c_0$.

Solution: The first-order condition is that

$$u'(c_0) = \delta \mathsf{E}[u'((w_0 - c_0)R_f + \tilde{y})]\,.$$

By Jensen's inequality and the convexity of u',

$$\mathsf{E}[u'((w_0 - c_0)R_f + \tilde{y})] > u'\left(\mathsf{E}[(w_0 - c_0)R_f + \tilde{y}]\right) = u'((w_0 - c_0)R_f)\,.$$

Thus,

$$u'(c_0) > \delta u'((w_0 - c_0)R_f)\,.$$

The first-order condition if $\tilde{y} = 0$ is for these to be equal. Because the left-hand side is decreasing in c_0 and the right-hand side increasing in c_0, equality requires that c_0 be increased. Thus, the optimal c_0 would be larger if $\tilde{y} = 0$.

2.9. Letting c_0^* denote optimal consumption in the previous problem, define the precautionary premium π by

$$u'((w_0 - \pi - c_0^*)R_f) = \mathsf{E}[u'((w_0 - c_0^*)R_f + \tilde{y})]\,.$$

(a) Show that c_0^* would be the optimal consumption of the investor if he had no end-of-period endowment and had initial wealth $w_0 - \pi$.

Solution: The first-order condition is that

$$u'(c_0^*) = \delta \mathsf{E}[u'((w_0 - c_0^*)R_f + \tilde{y})]\,.$$

By the definition of the precautionary premium, this implies

$$u'(c_0^*) = \delta u'((w_0 - \pi - c_0^*)R_f)\,.$$

This is the first-order condition for initial wealth $w_0 - \pi$ when $\tilde{y} = 0$.

(b) Assume the investor has CARA utility. Show that the precautionary premium is independent of initial wealth (again, no wealth effects with CARA utility).

Solution: With CARA utility $-\mathrm{e}^{-\alpha w}$, the marginal utility is $\alpha \mathrm{e}^{-\alpha w}$. Therefore the precautionary premium is π satisfying

$$\alpha \mathrm{e}^{-\alpha(w_0-\pi-c_0^*)R_f} = \alpha \mathsf{E}\left[\mathrm{e}^{-\alpha\left((w_0-c_0^*)R_f+\tilde{y}\right)}\right] .$$

Multiplying by $\mathrm{e}^{\alpha(w_0-c_0^*)R_f}/\alpha$ yields

$$\mathrm{e}^{\alpha\pi R_f} = \mathsf{E}\left[\mathrm{e}^{-\alpha\tilde{y}}\right] ,$$

with solution

$$\pi = \frac{1}{\alpha R_f}\log \mathsf{E}\left[\mathrm{e}^{-\alpha\tilde{y}}\right] .$$