first the special cases (i) risk tolerance $= A$ and (ii) risk tolerance $= Bw$. In case (i) use the fact that

$$\frac{u''(w)}{u'(w)} = \frac{\mathrm{d}\log u'(w)}{\mathrm{d}w}$$

and in case (ii) use the fact that

$$\frac{wu''(w)}{u'(w)} = \frac{\mathrm{d}\log u'(w)}{\mathrm{d}\log w}$$

to derive formulas for $\log u'(w)$ and hence $u'(w)$ and hence $u(w)$. For the case $A \neq 0$ and $B \neq 0$, define

$$v(w) = u\left(\frac{w-A}{B}\right),$$

show that the risk tolerance of v is Bw, apply the results from case (ii) to v, and then derive the form of u.

Solution: In case (i), set $\alpha = 1/A$. For any constant y,

$$\begin{aligned}
\log u'(w) &= \log u'(y) + \int_y^w \frac{\mathrm{d}\log u'(x)}{\mathrm{d}x}\,\mathrm{d}x \\
&= \log u'(y) + -\alpha\int_y^w \mathrm{d}x \\
&= \log u'(y) - \alpha(w-y)\,.
\end{aligned}$$

Hence,

$$u'(w) = u'(y)\mathrm{e}^{-\alpha(w-y)} = u'(y)\mathrm{e}^{\alpha y}\mathrm{e}^{-\alpha w}\,.$$

This implies

$$\begin{aligned}
u(w) &= u(y) + \int_y^w u'(x)\,\mathrm{d}x \\
&= u(y) + u'(y)\mathrm{e}^{\alpha y}\int_y^w \mathrm{e}^{-\alpha x}\,\mathrm{d}x \\
&= u(y) + u'(y)\mathrm{e}^{\alpha y}\frac{1}{\alpha}\left[\mathrm{e}^{-\alpha y} - \mathrm{e}^{-\alpha w}\right]\,.
\end{aligned}$$

This is an affine transform of $-\mathrm{e}^{-\alpha w}$. For u to be monotone, it must be a monotone affine transform of $-\mathrm{e}^{-\alpha w}$.

In case (ii), set $\rho = 1/B$. For any constant $y > 0$ and any $w > 0$,

$$\begin{aligned}\log u'(w) &= \log u'(y) + \int_y^w \frac{\mathrm{d}\log u'(x)}{\mathrm{d}\log x}\,\mathrm{d}\log x \\ &= \log u'(y) - \rho \int_y^w \mathrm{d}\log x \\ &= \log u'(y) - \rho(\log w - \log y)\,.\end{aligned}$$

Hence,

$$u'(w) = u'(y)\mathrm{e}^{-\rho(\log w - \log y)} = u'(y)y^{\rho}w^{-\rho}\,.$$

This implies

$$\begin{aligned}u(w) &= u(y) + \int_y^w u'(x)\,\mathrm{d}x \\ &= u(y) + u'(y)y^{\rho}\int_y^w x^{-\rho}\,\mathrm{d}x\,.\end{aligned}$$

If $\rho = 1$, then

$$u(w) = u(y) + u'(y)y(\log w - \log y)\,,$$

which is a monotone affine transform of $\log w$. If $\rho \neq 1$, then

$$u(w) = u(y) + u'(y)y^{\rho}\frac{1}{1-\rho}\left(w^{1-\rho} - y^{1-\rho}\right)\,.$$

which is an affine transform of $w^{1-\rho}/(1-\rho)$. For u to be monotone, it must be a monotone affine transform of $w^{1-\rho}/(1-\rho)$.

For the case $A \neq 0$ and $B \neq 0$, set

$$v(x) = u\left(\frac{x-A}{B}\right)$$

for $x > 0$. This implies

$$
\begin{aligned}
-\frac{v'(x)}{v''(x)} &= -B\frac{u'\left(\frac{x-A}{B}\right)}{u''\left(\frac{x-A}{B}\right)} \\
&= B\left[A + B\left(\frac{x-A}{B}\right)\right] \\
&= Bx\,.
\end{aligned}
$$

Therefore, from case (ii), on the region $x > 0$, either $v(x) = \log x$ if $B = 1$, or $v(x) = x^{1-\rho}/(1-\rho)$ for $\rho = 1/B$, up to an affine transform. Moreover,

$$
u(w) = v(A + Bw)\,.
$$

Hence, for w such that $A + Bw > 0$, either $u(w) = \log(A + Bw)$ if $B = 1$, or $u(w) = (A + Bw)^{1-\rho}/(1-\rho)$ for $\rho = 1/B$, up to an affine transform. Setting $\zeta = -A/B$, we have, up to an affine transform, $u(w) = \log(w - \zeta)$ on the region $w > \zeta$ if $B = 1$, or

$$
u(w) = \frac{1}{1-\rho}\left(\frac{w-\zeta}{\rho}\right)^{1-\rho}\,,
$$

on the region $(w - \zeta)/\rho > 0$. Monotonicity of u in the case $B \neq 1$ requires that u be a monotone affine transform of

$$
\frac{\rho}{1-\rho}\left(\frac{w-\zeta}{\rho}\right)^{1-\rho}\,.
$$

1.13. Show that risk neutrality [$u(w) = w$ for all w] can be regarded as a limiting case of negative exponential utility as $\alpha \to 0$ by showing that there are monotone affine transforms of negative exponential utility that converges to w as $\alpha \to 0$. Hint: Take an exact first-order Taylor series expansion of negative exponential utility, expanding in α around $\alpha = 0$. Writing the expansion as $c_0 + c_1\alpha$, show that

$$
\frac{-\mathrm{e}^{-\alpha w} - c_0}{\alpha} \to w
$$

as $\alpha \to 0$.

Solution: Set $f(\alpha) = -\mathrm{e}^{-\alpha w}$. We have

$$f(\alpha) = f(0) + f'(\hat{\alpha})\alpha$$

for some $0 < \hat{\alpha} < \alpha$, and $f'(\alpha) = w\mathrm{e}^{-\alpha w}$. Thus,

$$-\mathrm{e}^{-\alpha w} = -1 + w\mathrm{e}^{-\hat{\alpha} w}\alpha\,.$$

This implies

$$\frac{-\mathrm{e}^{-\alpha w} + 1}{\alpha} = w\mathrm{e}^{-\hat{\alpha} w} \to w\,,$$

as $\alpha \to 0$.

Chapter 2

Portfolio Choice

2.1. Suppose there is a risk-free asset with $R_f = 1.05$ and three risky assets each of which has an expected return equal to 1.10. Suppose the covariance matrix of the risky asset returns is

$$\Sigma = \begin{pmatrix} 0.09 & 0.06 & 0 \\ 0.06 & 0.09 & 0 \\ 0 & 0 & 0.09 \end{pmatrix}.$$

Suppose the returns are normally distributed. What is the optimal fraction of wealth to invest in each of the risky assets for a CARA investor with $\alpha w_0 = 2$? Why is the optimal investment higher for the third asset than for the other two?

Solution: The optimal portfolio of a CARA investor with multiple risky assets can be calculated by the formula (2.22). The optimal fraction of initial wealth is:

$$
\begin{aligned}
\pi &= \frac{1}{\alpha w_0}\Sigma^{-1}(\mu - R_f\iota) \\
&= \frac{1}{2}\begin{pmatrix} 0.09 & 0.06 & 0 \\ 0.06 & 0.09 & 0 \\ 0 & 0 & 0.09 \end{pmatrix}^{-1}\left[\begin{pmatrix} 1.10 \\ 1.10 \\ 1.10 \end{pmatrix} - \begin{pmatrix} 1.05 \\ 1.05 \\ 1.05 \end{pmatrix}\right] = \begin{pmatrix} 3.5 \\ 3.5 \\ 5.83 \end{pmatrix}
\end{aligned}
$$

The risky assets have the same variance and the same expected return, but the first two assets are positively correlated. Lower risk is achieved by holding more of the third asset, which is uncorrelated with the others.

2.2. Suppose there is a risk-free asset and n risky assets with payoffs $\tilde{x}_i$ and prices p_i. Assume the vector $\tilde{x} = (\tilde{x}_1 \cdots \tilde{x}_n)'$ is normally distributed with mean μ_x and nonsingular covariance matrix Σ_x. Let $p = (p_1 \cdots p_n)'$. Suppose there is consumption at date 0 and consider an investor with initial wealth w_0 and CARA utility at date 1:

$$u_1(c) = -\mathrm{e}^{-\alpha c}\,.$$

Let θ_i denote the number of shares the investor considers holding of asset i and set $\theta = (\theta_1 \cdots \theta_n)'$. The investor chooses consumption c_0 at date 0 and a portfolio θ, producing wealth $(w_0 - c_0 - \theta' p)R_f + \theta'\tilde{x}$ at date 1.

(a) Show that the optimal vector of share holdings is

$$\theta = \frac{1}{\alpha}\Sigma_x^{-1}(\mu_x - R_f p)\,.$$

Solution: The investor chooses date–0 consumption c_0 and a portfolio θ of risky assets to maximize

$$u(c_0) - \exp\left(-\alpha(w_0 - c_0 - \theta' p)R_f - \alpha\theta'\mu_x + \frac{1}{2}\alpha^2\theta'\Sigma_x\theta\right)\,.$$

The optimal portfolio θ is the portfolio that maximizes

$$-R_f p'\theta + \mu_x'\theta - \frac{1}{2}\alpha\theta'\Sigma_x\theta\,.$$

The first-order condition is

$$-R_f p + \mu_x - \alpha\Sigma_x\theta = 0\,,$$

with solution

$$\theta = \frac{1}{\alpha}\Sigma_x^{-1}(\mu_x - R_f p)\,.$$

(b) Suppose all of the asset prices are positive, so we can define returns $\tilde{x}_i/p_i$. Explain why (2.33) implies (2.22). Note: This is another illustration of the absence of wealth effects. Neither date-0 wealth nor date-0 consumption affects the optimal portfolio for a CARA investor.

Solution: Let P denote the $n \times n$ diagonal matrix with the ith diagonal element being p_i. Then $\phi = P\theta$, $\mu - R_f\iota = P^{-1}(\mu_x - R_f p)$, and $\Sigma = P^{-1}\Sigma_x P^{-1}$. Therefore, multiplying (2.33) by P gives

$$\phi = \frac{1}{\alpha}P(P\Sigma P)^{-1}P(\mu - R_f\iota) = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f\iota)\,.$$

2.3. Suppose there is a risk-free asset with return R_f and a risky asset with return $\tilde{R}$. Consider an investor who maximizes expected end-of-period utility of wealth and who has CARA utility and invests w_0. Suppose the investor has a random endowment $\tilde{y}$ at the end of the period, so his end-of-period wealth is $\phi_f R_f + \phi\tilde{R} + \tilde{y}$, where ϕ_f denotes the investment in the risk-free asset and ϕ the investment in the risky asset.

(a) Suppose $\tilde{y}$ and $\tilde{R}$ are independent. Show that the optimal ϕ is the same as if there were no end-of-period endowment. Hint: Use the law of iterated expectations as in Section 1.5 and the fact that if $\tilde{v}$ and $\tilde{x}$ are independent random variables then $\mathsf{E}[\tilde{v}\tilde{x}] = \mathsf{E}[\tilde{v}]\mathsf{E}[\tilde{x}]$.

Solution: The expected utility is

$$-\mathsf{E}\left[\exp\left(-\alpha w_0 R_f - \alpha\phi\left(\tilde{R} - R_f\right) - \alpha\tilde{y}\right)\right] = -\mathsf{E}\left[\mathrm{e}^{-\alpha w_0 R_f - \alpha\phi\left(\tilde{R}-R_f\right)}\mathrm{e}^{-\alpha\tilde{y}}\right].$$

By independence, this equals

$$-\mathsf{E}\left[\mathrm{e}^{-\alpha w_0 R_f - \alpha\phi\left(\tilde{R}-R_f\right)}\right]\mathsf{E}\left[\mathrm{e}^{-\alpha\tilde{y}}\right],$$

and maximizing this over ϕ is equivalent to maximizing

$$-\mathsf{E}\left[\mathrm{e}^{-\alpha w_0 R_f - \alpha\phi\left(\tilde{R}-R_f\right)}\right],$$

which is the same as if $\tilde{y} = 0$.

(b) Define $b = \operatorname{cov}(\tilde{y}, \tilde{R}) / \operatorname{var}(\tilde{R})$, $a = (\mathsf{E}[\tilde{y}] - b\mathsf{E}[\tilde{R}])/R_f$ and $\tilde{\varepsilon} = \tilde{y} - aR_f - b\tilde{R}$. Show that $\tilde{y} = aR_f + b\tilde{R} + \tilde{\varepsilon}$ and that $\tilde{\varepsilon}$ has a zero mean and is uncorrelated with $\tilde{R}$. Note: This is an example of an orthogonal projection, which is discussed in more generality in Section 3.5.

Solution: From the definition of $\tilde{\varepsilon}$, we have $\tilde{y} = aR_f + b\tilde{R} + \tilde{\varepsilon}$. We need to show that $\tilde{\varepsilon}$ has a zero mean and is uncorrelated with $\tilde{R}$. We have

$$\mathsf{E}[\varepsilon] = \mathsf{E}[\tilde{y} - aR_f - b\tilde{R}] = \mathsf{E}[\tilde{y}] - [(\mathsf{E}[\tilde{y}] - b\mathsf{E}[\tilde{R}])/R_f].R_f - b\mathsf{E}[\tilde{R}] = 0\,.$$

Furthermore,

$$\begin{aligned}
\operatorname{cov}(\tilde{\varepsilon}, \tilde{R}) &= \operatorname{cov}(\tilde{y} - aR_f - b\tilde{R}, \tilde{R}) \\
&= \operatorname{cov}(\tilde{y}, \tilde{R}) - b\operatorname{var}(\tilde{R}) \\
&= 0\,,
\end{aligned}$$

using the definition of b for the last equality.

(c) Suppose $\tilde{y}$ and $\tilde{R}$ have a joint normal distribution. Using the result of the previous part, show that the optimal ϕ is $\phi^* - b$, where ϕ^* denotes the optimal investment in the risky asset when there is no end-of-period endowment.

Solution: The expected end-of-period wealth is

$$w_0 R_f + \phi(\mathsf{E}[\tilde{R}] - R_f + \mathsf{E}[\tilde{y}] = (w_0 + a - \phi)R_f + (\phi + b)\mathsf{E}[\tilde{R}]\,,$$

and the variance of end-of-period wealth is

$$\phi^2 \operatorname{var}(\tilde{R}) + 2\phi \operatorname{cov}(\tilde{R}, \tilde{y}) + \operatorname{var}(\tilde{y}) = (\phi^2 + 2\phi b + b^2) \operatorname{var}(\tilde{R}) + \operatorname{var}(\tilde{\varepsilon})\,.$$

The expected utility is

$$-\exp\left(-\alpha\left((w_0 + a - \phi)R_f + (\phi + b)\mathsf{E}[\tilde{R}]\right) + \frac{1}{2}\alpha^2\left((\phi^2 + 2\phi b + b^2)\operatorname{var}(\tilde{R}) + \operatorname{var}(\tilde{\varepsilon})\right)\right)\,.$$

Maximizing this over ϕ is equivalent to maximizing

$$\phi(\mathsf{E}[\tilde{R}] - R_f) - \frac{1}{2}\alpha(\phi^2 + 2\phi b)\operatorname{var}(\tilde{R})\,,$$

for which the solution is

$$\phi = \frac{\mathsf{E}[\tilde{R}] - R_f}{\alpha \operatorname{var}(\tilde{R})} - b\,.$$

2.4. Consider a CARA investor with n risky assets having normally distributed returns, as studied in Section 2.4, but suppose there is no risk-free asset, so the budget constraint is $\iota'\phi = w_0$. Show that the optimal portfolio is

$$\phi = \frac{1}{\alpha}\Sigma^{-1}\mu + \left(\frac{\alpha w_0 - \iota'\Sigma^{-1}\mu}{\alpha\iota'\Sigma^{-1}\iota}\right)\Sigma^{-1}\iota\,.$$

Note: As will be seen in Section 5.2, the two vectors $\Sigma^{-1}\mu$ and $\Sigma^{-1}\iota$ play an important role in mean-variance analysis even without the CARA/normal assumption.

Solution: The expected payoff of a portfolio ϕ is $\phi'\mu$ and the variance is $\phi'\Sigma\phi$. The expected utility is

$$-\exp\left(-\alpha\phi'\mu + \frac{1}{2}\alpha^2\phi'\Sigma\phi\right)\,.$$

Maximizing this is equivalent to maximizing

$$\phi'\mu - \frac{1}{2}\alpha\phi'\Sigma\phi\,.$$

Let λ denote the Lagrange multiplier for the constraint $\iota'\phi = w_0$. The Lagrangean is

$$\phi'(\mu - \lambda\iota) - \frac{1}{2}\alpha\phi'\Sigma\phi\,,$$

and the first-order condition is

$$\mu - \lambda\iota - \alpha\Sigma\phi = 0\,,$$

which is solved by

$$\phi = \frac{1}{\alpha}\Sigma^{-1}(\mu - \lambda\iota)\,.$$

Imposing the constraint $\iota'\phi = w_0$ yields

$$\frac{1}{\alpha}\iota'\Sigma^{-1}\mu - \frac{\lambda}{\alpha}\iota'\Sigma^{-1}\iota = w_0\,.$$

Therefore,

$$\lambda = \frac{\iota'\Sigma^{-1}\mu - \alpha w_0}{\iota'\Sigma^{-1}\iota}\,,$$

and

$$\phi = \frac{1}{\alpha}\Sigma^{-1}\mu + \left(\frac{\alpha w_0 - \iota'\Sigma^{-1}\mu}{\alpha\iota'\Sigma^{-1}\iota}\right)\Sigma^{-1}\iota\,.$$

2.5. Suppose there is a risk-free asset and n risky assets. Consider an investor with quadratic utility who seeks to maximize

$$\zeta\mathsf{E}[\tilde{w}] - \frac{1}{2}\mathsf{E}[\tilde{w}]^2 - \frac{1}{2}\,\mathrm{var}(\tilde{w})\,.$$

Show that the optimal portfolio for the investor is

$$\phi = \frac{1}{1+\kappa^2}(\zeta - w_0R_f)\Sigma^{-1}(\mu - R_f\iota)\,,$$

where

$$\kappa^2 = (\mu - R_f\iota)'\Sigma^{-1}(\mu - R_f\iota)\,.$$

Hint: In the first-order conditions, define $\gamma = (\mu - R_f\iota)'\phi$, solve for ϕ in terms of γ, and then compute γ. Note: We will see in Chapter 5 that κ is the maximum Sharpe ratio of any portfolio.

Solution: The expected payoff of a portfolio ϕ of risky assets is $w_0R_f + \phi'(\mu - R_f\iota)$, and the variance is $\phi'\Sigma\phi$. The expected utility is

$$\begin{aligned}
&\zeta[w_0R_f + \phi'(\mu - R_f\iota)] - \frac{1}{2}[w_0R_f + \phi'(\mu - R_f\iota)]^2 - \frac{1}{2}\phi'\Sigma\phi \\
&= \zeta[w_0R_f + \phi'(\mu - R_f\iota)] - \frac{1}{2}w_0^2R_f^2 - w_0R_f\phi'(\mu - R_f\iota) - \frac{1}{2}\phi'(\mu - R_f\iota)(\mu - R_f\iota)'\phi - \frac{1}{2}\phi'\Sigma\phi\,.
\end{aligned}$$

The first-order condition for maximizing this is

$$\zeta(\mu - R_f\iota) - w_0R_f(\mu - R_f\iota) - (\mu - R_f\iota)(\mu - R_f\iota)'\phi - \Sigma\phi = 0\,.$$

Setting $\gamma = (\mu - R_f\iota)'\phi$, we have

$$\zeta(\mu - R_f\iota) - w_0R_f(\mu - R_f\iota) - \gamma(\mu - R_f\iota) - \Sigma\phi = 0\,,$$

with solution

$$\phi = (\zeta - w_0R_f - \gamma)\Sigma^{-1}(\mu - R_f\iota)\,.$$

Thus,

$$\begin{aligned}
\gamma &= (\zeta - w_0R_f - \gamma)(\mu - R_f\iota)'\Sigma^{-1}(\mu - R_f\iota) \\
&= (\zeta - w_0R_f - \gamma)\kappa^2\,,
\end{aligned}$$

implying

$$\gamma = \frac{\kappa^2}{1+\kappa^2}(\zeta - w_0R_f)\,,$$

and

$$\phi = \frac{1}{1+\kappa^2}(\zeta - w_0 R_f)\Sigma^{-1}(\mu - R_f \iota)\,.$$

2.6. Consider a utility function $v(c_0, c_1)$. The marginal rate of substitution (MRS) is defined to be the negative of the slope of an indifference curve and is equal to

$$\text{MRS}(c_0, c_1) = \frac{\partial v(c_0, c_1)/\partial c_0}{\partial v(c_0, c_1)/\partial c_1}\,.$$

The elasticity of intertemporal substitution is defined as

$$\frac{\mathrm{d}\log(c_1/c_0)}{\mathrm{d}\log \text{MRS}(c_0, c_1)}\,,$$

where the marginal rate of substitution is varied holding utility constant. Show that, if

$$v(c_0, c_1) = \frac{1}{1-\rho}c_0^{1-\rho} + \frac{\delta}{1-\rho}c_1^{1-\rho}\,,$$

then the EIS is $1/\rho$.

Solution: Holding utility constant implies

$$c_0^{-\rho}\,\mathrm{d}c_0 + \delta c_1^{-\rho}\,\mathrm{d}c_1 = 0\,,$$

so

$$-\frac{\mathrm{d}c_1}{\mathrm{d}c_0} = \frac{1}{\delta}\left(\frac{c_0}{c_1}\right)^{-\rho}\,.$$

This is the marginal rate of substitution. Setting $x = c_1/c_0$, we have

$$\log \text{MRS} = -\log\delta + \rho\log x\,.$$

Hence,

$$\frac{\mathrm{d}\log \text{MRS}}{\mathrm{d}\log x} = \rho\,.$$

The elasticity of intertemporal substitution is the reciprocal $1/\rho$.