Chapter 1

Utility and Risk Aversion

1.1. Calculate the risk tolerance of each of the LRT utility functions (negative exponential, log, power, shifted log, and shifted power) to verify the formulas for risk tolerance given in Section 1.3

$$u(w) = -\mathrm{e}^{-\alpha w} \quad \Rightarrow \quad u'(w) = \alpha \mathrm{e}^{-\alpha w}\,, \quad u''(w) = -\alpha^2 \mathrm{e}^{-\alpha w}\,, \quad -\frac{u'(w)}{u''(w)} = \frac{1}{\alpha}\,.$$

$$u(w) = \log w \quad \Rightarrow \quad u'(w) = \frac{1}{w}\,, \quad u''(w) = -\frac{1}{w^2}\,, \quad -\frac{u'(w)}{u''(w)} = w\,.$$

$$u(w) = \frac{1}{1-\rho} w^{1-\rho} \quad \Rightarrow \quad u'(w) = w^{-\rho}\,, \quad u''(w) = -\rho w^{-\rho-1}\,, \quad -\frac{u'(w)}{u''(w)} = \frac{w}{\rho}\,.$$

$$u(w) = \log(w-\zeta) \quad \Rightarrow \quad u'(w) = \frac{1}{w-\zeta}\,, \quad u''(w) = -\frac{1}{(w-\zeta)^2}\,, \quad -\frac{u'(w)}{u''(w)} = w-\zeta\,.$$

$$u(w) = \frac{\rho}{1-\rho}\left(\frac{w-\zeta}{\rho}\right)^{1-\rho} \quad \Rightarrow \quad u'(w) = \left(\frac{w-\zeta}{\rho}\right)^{-\rho}\,, \quad u''(w) = -\left(\frac{w-\zeta}{\rho}\right)^{-\rho-1}\,,$$

$$-\frac{u'(w)}{u''(w)} = \frac{w-\zeta}{\rho}\,.$$

1.2. Consider a person with constant relative risk aversion ρ who has wealth w.

(a) Suppose he faces a gamble in which he wins or loses some amount x with equal probabilities. Derive a formula for the amount π that he would pay to avoid the gamble; that is, find π

satisfying

$$u(w-\pi) = \frac{1}{2}u(w-x) + \frac{1}{2}u(w+x)$$

when u is log or power utility.

Solution: When u is log utility, we have

$$\log(w-\pi) = \frac{1}{2}\log(w-x) + \frac{1}{2}\log(w+x) = \log\sqrt{(w-x)(w+x)}\,.$$

Exponentiating both sides and rearranging gives

$$\pi = w - \sqrt{(w-x)(w+x)}\,.$$

For power utility, we have

$$\frac{1}{1-\rho}(w-\pi)^{1-\rho} = \frac{1}{2}\cdot\frac{1}{1-\rho}(w-x)^{1-\rho} + \frac{1}{2}\cdot\frac{1}{1-\rho}(w+x)^{1-\rho}\,.$$

This implies

$$\pi = w - \left[\frac{1}{2}(w-x)^{1-\rho} + \frac{1}{2}(w+x)^{1-\rho}\right]^{1/(1-\rho)}\,.$$

(b) Suppose instead that he is offered a gamble in which he loses x or wins y with equal probabilities. Find the maximum possible loss x at which he would accept the gamble; that is, find x satisfying

$$u(w) = \frac{1}{2}u(w-x) + \frac{1}{2}u(w+y)$$

when u is log or power utility.

Solution: When u is log utility, we have

$$\log w = \frac{1}{2}\log(w-x) + \frac{1}{2}\log(w+y) = \log\sqrt{(w-x)(w+y)}\,.$$

Exponentiating and then squaring both sides and rearranging gives

$$x = \frac{wy}{w+y}\,.$$

For power utility, we have

$$\frac{1}{1-\rho}w^{1-\rho} = \frac{1}{2}\cdot\frac{1}{1-\rho}(w-x)^{1-\rho} + \frac{1}{2}\cdot\frac{1}{1-\rho}(w+y)^{1-\rho}\,.$$

This implies

$$x = w - \left[2w^{1-\rho} - (w+y)^{1-\rho}\right]^{1/(1-\rho)}\,.$$

(c) Suppose the person has wealth of \$100,000 and faces a gamble as in Part (a). Use the answer in Part (a) to calculate the amount he would pay to avoid the gamble, for various values of ρ (say, between 0.5 and 40), and for $x = \$100$, $x = \$1{,}000$, $x = \$10{,}000$, and $x = \$25{,}000$. For large gambles, do large values of ρ seem reasonable? What about small gambles?

Solution:

ρ	$x = \$100$	$x = \$1,000$	$x = \$10,000$	$x = \$25,000$
0.5	\$0.03	\$2.50	\$251	\$1,588
1	\$0.05	\$5	\$501	\$3,175
2	\$0.10	\$10	\$1,000	\$6,250
5	\$0.25	\$25	\$2,434	\$13,486
10	\$0.50	\$50	\$4,424	\$19,086
15	\$0.75	\$75	\$5,826	\$21,198
20	\$1.00	\$99	\$6,763	\$22,214
30	\$1.50	\$148	\$7,832	\$23,186
40	\$2.00	\$195	\$8,387	\$23,655

For the largest gamble, $\rho > 5$ (or, perhaps $\rho > 2$) would seem unreasonable. But, for $\rho \leq 5$, the premium for the \$100 gamble is \$0.25 or less, which may be too small.

(d) Suppose $\rho > 1$, and the person is offered a gamble as in Part (b). Show that he will reject the gamble no matter how large y is if

$$\frac{x}{w} \geq \; 1 - 0.5^{1/(\rho-1)} \qquad \Leftrightarrow \qquad \rho \geq \frac{\log(0.5) + \log(1 - x/w)}{\log(1 - x/w)}\,.$$

For example, with wealth of \$100,000, the person would reject a gamble in which he loses \$10,000 or wins 1 trillion dollars with equal probabilities when ρ satisfies this inequality for $x/w = 0.1$. What values of ρ (if any) seem reasonable?

Solution: Given $1 - \rho < 0$, the person rejects the gamble if

$$w^{1-\rho} < 0.5(w-x)^{1-\rho} + 0.5(w+y)^{1-\rho}\,.$$

This is true for all $y > 0$ if

$$\begin{aligned}
w^{1-\rho} \leq 0.5(w-x)^{1-\rho} \quad &\Leftrightarrow \quad w \geq 0.5^{\frac{1}{1-\rho}}(w-x) \\
&\Leftrightarrow \quad 0.5^{\frac{1}{1-\rho}} x \geq \left[0.5^{\frac{1}{1-\rho}} - 1\right] w \\
&\Leftrightarrow \quad \frac{x}{w} \geq 1 - 0.5^{\frac{1}{\rho-1}} \\
&\Leftrightarrow \quad 0.5^{\frac{1}{\rho-1}} \geq 1 - \frac{x}{w} \\
&\Leftrightarrow \quad \frac{1}{\rho-1}\log(0.5) \geq \log(1 - x/w) \\
&\Leftrightarrow \quad \frac{1}{\rho-1} \leq \frac{\log(1-x/w)}{\log(0.5)} \\
&\Leftrightarrow \quad \rho - 1 \geq \frac{\log(0.5)}{\log(1-x/w)} \\
&\Leftrightarrow \quad \rho \geq \frac{\log(0.5) + \log(1-x/w)}{\log(1-x/w)}
\end{aligned}$$

Thus, all gambles involving 1% losses are rejected if $\rho \geq 70$, 2% losses if $\rho \geq 36$, 10% losses if $\rho \geq 7.6$, 25% losses if $\rho \geq 3.5$, and 50% losses if $\rho \geq 2$. Surely, there should be some possible gain that would compensate someone for a 50% chance of a 10% loss, implying $\rho < 7.6$. One could obviously argue for even smaller ρ.

1.3. This exercise is a very simple version of a model of the bid-ask spread presented by Stoll (1978). Consider an individual with constant absolute risk aversion α. Assume $\tilde{w}$ and $\tilde{x}$ are joint normally distributed with means μ_w and μ_x, variances σ_w^2 and σ_x^2 and correlation coefficient ρ.

(a) Compute the maximum amount the individual would pay to obtain $\tilde{w}$ when starting with $\tilde{x}$; that is, compute BID satisfying

$$\mathsf{E}[u(\tilde{x})] = \mathsf{E}[u(\tilde{x} + \tilde{w} - \text{BID})]\,.$$

Solution: Note that

$$\mathsf{E}[u(\tilde{w})] = -\exp\left(-\alpha\mathsf{E}[\tilde{w}] + \frac{1}{2}\alpha^2\operatorname{var}(\tilde{w})\right)\,.$$

We have

$$\mathsf{E}[u(\tilde{w}{+}\tilde{x}{-}\text{BID})] = -\exp\left(-\alpha\mathsf{E}[\tilde{w}] - \alpha\mathsf{E}[\tilde{x}] + \alpha\text{BID} + \frac{1}{2}\alpha^2[\operatorname{var}(\tilde{w}) + 2\operatorname{cov}(\tilde{x},\tilde{w}) + \operatorname{var}(\tilde{x})]\right)\,.$$

Thus, BID satisfies

$$1 = \exp\left(-\alpha\mathsf{E}[\tilde{x}] + \alpha\text{BID} + \frac{1}{2}\alpha^2[2\operatorname{cov}(\tilde{x},\tilde{w}) + \operatorname{var}(\tilde{x})]\right)\,.$$

This implies

$$\text{BID} = \mathsf{E}[\tilde{x}] - \alpha\operatorname{cov}(\tilde{x},\tilde{w}) - \frac{1}{2}\alpha\operatorname{var}(\tilde{x}) = \mu_x - \alpha\rho\sigma_x\sigma_w - \frac{1}{2}\alpha\sigma_x^2\,.$$

(b) Compute the minimum amount the individual would require to accept the payoff $-\tilde{w}$ when starting with $\tilde{x}$; that is, compute ASK satisfying

$$\mathsf{E}[u(\tilde{x})] = \mathsf{E}[u(\tilde{x} - \tilde{w} + \text{ASK})]\,.$$

Solution: We have

$$\mathsf{E}[u(\tilde{w}{-}\tilde{x}{+}\text{ASK})] = -\exp\left(-\alpha\mathsf{E}[\tilde{w}] + \alpha\mathsf{E}[\tilde{x}] - \alpha\text{ASK} + \frac{1}{2}\alpha^2[\operatorname{var}(\tilde{w}) - 2\operatorname{cov}(\tilde{x},\tilde{w}) + \operatorname{var}(\tilde{x})]\right)\,.$$

Thus, ASK satisfies

$$1 = \exp\left(\alpha \mathsf{E}[\tilde{x}] - \alpha \text{ASK} + \frac{1}{2}\alpha^2[-2\operatorname{cov}(\tilde{x}, \tilde{w}) + \operatorname{var}(\tilde{x})]\right) .$$

This implies

$$\text{ASK} = \mathsf{E}[\tilde{x}] - \alpha \operatorname{cov}(\tilde{x}, \tilde{w}) + \frac{1}{2}\alpha \operatorname{var}(\tilde{x}) = \mu_x - \alpha\rho\sigma_x\sigma_w + \frac{1}{2}\alpha\sigma_x^2 .$$

Note that the bid-ask spread is ASK $-$ BID $= \alpha\sigma_x^2$.

1.4. Calculate the mean, variance, and skewness of the following two random variables:

$$\tilde{w}_1 = \begin{cases} 2.45 & \text{with probability } 0.5141\,, \\ 7.49 & \text{with probability } 0.4859\,, \end{cases}$$

$$\tilde{w}_2 = \begin{cases} 0 & \text{with probability } 0.12096\,, \\ 4.947 & \text{with probability } 0.750085\,, \\ 10 & \text{with probability } 0.128955\,. \end{cases}$$

You should see that $\tilde{w}_2$ has a higher mean, lower variance, and higher skewness than $\tilde{w}_1$. Show that, nevertheless, $\tilde{w}_1$ is preferred to $\tilde{w}_2$ by a CARA investor with absolute risk aversion equal to 1, by a CRRA investor with relative risk aversion equal to 1/2, and by an investor with shifted log utility $\log(1 + w)$.

Solution: Let μ_i, σ_i^2 and γ_i be the mean, variance, and skewness of the random variable $\tilde{w}_i$:

$\mu_1 = 4.8989$, $\sigma_1^2 = 6.3453$, $\gamma_1 = 0.0564$

$\mu_2 = 5.0002$, $\sigma_2^2 = 6.2410$, $\gamma_2 = 0.0637$

CARA investor with absolute risk aversion equal to 1: $u(w) = -e^{-w}$, so $E[u(\tilde{w}_1)] = -0.0446$ and $E[u(\tilde{w}_2)] = -0.1263$

CRRA investor with relative risk aversion equal to 1/2: $u(w) = 2w^{0.5}$, so $E[u(\tilde{w_1})] = 4.2690$ and $E[u(\tilde{w_2})] = 4.1522$

An investor with shifted log utility: $u(w) = log(1+w)$, so $E[u(\tilde{w_1})] = 0.7278$ and $E[u(\tilde{w_2})] = 0.7151$

1.5. Consider a person with constant relative risk aversion ρ.

(a) Verify that the fraction of wealth he will pay to avoid a gamble that is proportional to wealth is independent of initial wealth (that is, show that π defined in (1.15) is independent of w for logarithmic and power utility).

Solution: For log utility, the left-hand side of (1.15) is

$$\log((1-\pi)w) = \log(1-\pi) + \log w\,,$$

and the right-hand side is

$$\mathsf{E}[\log((1+\tilde{\varepsilon})w)] = \mathsf{E}[\log(1+\tilde{\varepsilon})] + \log w\,,$$

so (1.15) is equivalent to

$$\log(1-\pi) = \mathsf{E}[\log(1+\tilde{\varepsilon})] \quad \Leftrightarrow \quad \pi = 1 - \exp\left(\mathsf{E}[\log(1+\tilde{\varepsilon})]\right)\,.$$

Hence, π does not depend on w. For power utility, the left-hand side of (1.15) is

$$\frac{1}{1-\rho}((1-\pi)w)^{1-\rho} = \frac{1}{1-\rho}w^{1-\rho}(1-\pi)^{1-\rho}\,,$$

and the right-hand side is

$$\mathsf{E}\left[\frac{1}{1-\rho}((1+\tilde{\varepsilon})w)^{1-\rho}\right] = \frac{1}{1-\rho}w^{1-\rho}\mathsf{E}\left[(1+\tilde{\varepsilon})^{1-\rho}\right]\,,$$

so (1.15) is equivalent to

$$(1-\pi)^{1-\rho} = \mathsf{E}\left[(1+\tilde{\varepsilon})^{1-\rho}\right] \quad \Leftrightarrow \quad \pi = 1 - \left(\mathsf{E}\left[(1+\tilde{\varepsilon})^{1-\rho}\right]\right)^{\frac{1}{1-\rho}}\,.$$

Hence, π does not depend on w.

(b) Consider a gamble $\tilde{\varepsilon}$. Assume $1+\tilde{\varepsilon}$ is lognormally distributed; specifically, assume $1+\tilde{\varepsilon} = \mathrm{e}^{\tilde{z}}$, where $\tilde{z}$ is normally distributed with variance σ^2 and mean $-\sigma^2/2$. By the rule for means of exponentials of normals, $\mathsf{E}[\tilde{\varepsilon}] = 0$. Show that π defined in (1.15) equals

$$1 - \mathrm{e}^{-\rho\sigma^2/2}\,.$$

Note: This is consistent with the approximation (1.5), because a first-order Taylor series expansion of the exponential function e^x around $x = 0$ shows that $\mathrm{e}^x \approx 1 + x$ when $|x|$ is small.

Solution: We have $\mathsf{E}[\log(1+\tilde{\varepsilon})] = \mathsf{E}[\tilde{z}] = -\sigma^2/2$, so the proportional risk premium for log utility is

$$\pi = 1 - \mathrm{e}^{-\sigma^2/2}\,.$$

For $\rho \neq 1$,

$$\mathsf{E}\left[(1+\tilde{\varepsilon})^{1-\rho}\right] = \mathsf{E}\left[\mathrm{e}^{(1-\rho)\tilde{z}}\right] = \mathrm{e}^{-(1-\rho)\sigma^2/2+(1-\rho)^2\sigma^2/2} = \mathrm{e}^{-\rho(1-\rho)\sigma^2/2}\,.$$

Therefore, the proportional risk premium is

$$\pi = 1 - \mathrm{e}^{-\rho\sigma^2/2}\,.$$

1.6. Use the law of iterated expectations to show that if $\mathsf{E}[\tilde{\varepsilon}|\tilde{y}] = 0$ then $\mathrm{cov}(\tilde{y},\tilde{\varepsilon}) = 0$ (thus mean-independence implies uncorrelated).

Solution: By iterated expectations and mean-independence,

$$\mathsf{E}[\tilde{y}\tilde{\varepsilon}] = \mathsf{E}[\tilde{y}\mathsf{E}[\tilde{\varepsilon}|\tilde{y}]] = 0\,.$$

Furthermore,

$$\mathsf{E}[\tilde{\varepsilon}] = \mathsf{E}[\mathsf{E}[\tilde{\varepsilon}|\tilde{y}]] = 0\,.$$

Therefore,

$$\text{cov}(\tilde{y}, \tilde{\varepsilon}) = \mathsf{E}[\tilde{y}\tilde{\varepsilon}] - \mathsf{E}[\tilde{y}]\mathsf{E}[\tilde{\varepsilon}] = 0\,.$$

1.7. Let $\tilde{y} = \mathrm{e}^{\tilde{x}}$, where $\tilde{x}$ is normally distributed with mean μ and variance σ^2. Show that

$$\frac{\text{stdev}(\tilde{y})}{\mathsf{E}[\tilde{y}]} = \sqrt{\mathrm{e}^{\sigma^2} - 1}\,.$$

Solution: We have $\mathsf{E}[\tilde{y}] = \mathrm{e}^{\mu+\sigma^2/2}$, and

$$\begin{aligned}
\text{var}(\tilde{y}) &= \mathsf{E}[\tilde{y}^2] - \mathsf{E}[\tilde{y}]^2 \\
&= \mathsf{E}\left[\mathrm{e}^{2\tilde{x}}\right] - \mathrm{e}^{2(\mu+\sigma^2/2)} \\
&= \mathrm{e}^{2\mu+2\sigma^2} - \mathrm{e}^{2\mu+\sigma^2} \\
&= \mathrm{e}^{2\mu+\sigma^2}\left(\mathrm{e}^{\sigma^2} - 1\right) \\
&= \mathsf{E}[\tilde{y}]^2\left(\mathrm{e}^{\sigma^2} - 1\right)\,,
\end{aligned}$$

so

$$\text{stdev}(\tilde{y}) = \mathsf{E}[\tilde{y}]\sqrt{\mathrm{e}^{\sigma^2} - 1}\,.$$

1.8. The notation and concepts in this exercise are from Appendix A. Suppose there are three possible states of the world which are equally likely, so $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = \mathbb{P}(\{\omega_3\}) = 1/3$. Let $\mathcal{G}$ be the collection of all subsets of Ω:

$$\mathcal{G} = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\}\,.$$

Let $\tilde{x}$ and $\tilde{y}$ be random variables, and set $a_i = \tilde{x}(\omega_i)$ for $i = 1, 2, 3$. Suppose $\tilde{y}(\omega_1) = b_1$ and $\tilde{y}(\omega_2) = \tilde{y}(\omega_3) = b_2 \neq b_1$.

(a) What is $\text{prob}(\tilde{x} = a_j \mid \tilde{y} = b_i)$ for $i = 1, 2$ and $j = 1, 2, 3$?

Solution:

$$\text{prob}(\tilde{x} = a_1 \mid \tilde{y} = b_1) = 1\,,$$

$$\text{prob}(\tilde{x} = a_2 \mid \tilde{y} = b_1) = 0\,,$$

$$\text{prob}(\tilde{x} = a_3 \mid \tilde{y} = b_1) = 0\,,$$

$$\text{prob}(\tilde{x} = a_1 \mid \tilde{y} = b_2) = 0\,,$$

$$\text{prob}(\tilde{x} = a_2 \mid \tilde{y} = b_2) = 1/2\,,$$

$$\text{prob}(\tilde{x} = a_3 \mid \tilde{y} = b_2) = 1/2\,.$$

(b) What is $\mathsf{E}[\tilde{x} \mid \tilde{y} = b_i]$ for $i = 1, 2$?

Solution:

$$\mathsf{E}[\tilde{x} \mid \tilde{y} = b_1] = a_1\,,$$

$$\mathsf{E}[\tilde{x} \mid \tilde{y} = b_2] = (a_2 + a_3)/2\,.$$

(c) What is the σ–field generated by $\tilde{y}$?

Solution: The σ–field generated by $\tilde{y}$ is

$$\{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}\,.$$

1.9. Suppose an investor has log utility: $u(w) = \log w$ for each $w > 0$.

(a) Construct a gamble $\tilde{w}$ such that $\mathsf{E}[u(\tilde{w})] = \infty$. Verify that $\mathsf{E}[\tilde{w}] = \infty$.

Solution: Consider flipping a sequence of coins and having wealth e^{2^n} if the first heads appears on the n–th toss. The probability of the first heads appearing on the n–th toss is 2^{-n}, so the expected utility is

$$\sum_{n=1}^{\infty} 2^{-n} \log\left(\mathrm{e}^{2^n}\right) = \sum_{n=1}^{\infty} 1 = \infty\,.$$

(b) Construct a gamble $\tilde{w}$ such that $\tilde{w} > 0$ in each state of the world and $\mathsf{E}[u(\tilde{w})] = -\infty$.

Solution: Consider flipping coins and having wealth e^{-2^n} if the first heads appears on the n–th toss. The expected utility is

$$\sum_{n=1}^{\infty} 2^{-n} \log\left(\mathrm{e}^{-2^n}\right) = \sum_{n=1}^{\infty} -1 = -\infty\,.$$

(c) Given a constant wealth w, construct a gamble $\tilde{\varepsilon}$ with $w + \tilde{\varepsilon} > 0$ in each state of the world, $\mathsf{E}[\tilde{\varepsilon}] = 0$ and $\mathsf{E}[u(w + \tilde{\varepsilon})] = -\infty$.

Solution: Obviously, there are many ways to do this. Here is one. Let $0 < \delta < 1$ be such that

$$w > \frac{\delta}{1+\delta} \sum_{n=1}^{\infty} 2^{-n} \mathrm{e}^{-2^n}\,.$$

Define $p = \delta/(1+\delta)$. With probability $1-p$, let

$$\tilde{\varepsilon} = \delta\left(w - \sum_{n=1}^{\infty} 2^{-n} \mathrm{e}^{-2^n}\right)\,.$$

On this event, we have

$$w + \tilde{\varepsilon} = (1+\delta)w - \delta \sum_{n=1}^{\infty} 2^{-n} \mathrm{e}^{-2^n} > 0\,.$$

For $n = 1, 2, \ldots$, let

$$\tilde{\varepsilon} = \mathrm{e}^{-2^n} - w$$

with probability $p2^{-n}$. Then $w + \tilde{\varepsilon} > 0$ in each state of the world, and

$$\begin{aligned}
\mathsf{E}[\tilde{\varepsilon}] &= (1-p)\delta\left(w - \sum_{n=1}^{\infty} 2^{-n} \mathrm{e}^{-2^n}\right) + p\left[\sum_{n=1}^{\infty} 2^{-n}\left(\mathrm{e}^{-2^n} - w\right)\right] \\
&= [(1-p)\delta - p]\left(w - \sum_{n=1}^{\infty} 2^{-n} \mathrm{e}^{-2^n}\right) \\
&= 0\,.
\end{aligned}$$

Moreover,

$$\mathsf{E}[u(w+\tilde{\varepsilon})] = (1-p)\log\left((1+\delta)w - \delta\sum_{n=1}^{\infty} 2^{-n}\mathrm{e}^{-2^n}\right) + p\left[\sum_{n=1}^{\infty} 2^{-n}\log\left(\mathrm{e}^{-2^n}\right)\right] = -\infty\,.$$

1.10. Which LRT utility functions are DARA utility functions with increasing relative risk aversion, for some parameter values? Which of these utility functions are monotone increasing and bounded on the domain $w \geq 0$?

Solution: Shifted log and shifted power utility functions have absolute risk aversion $\rho/(w-\zeta)$ and relative risk aversion $\rho w/(w-\zeta)$, with $\rho = 1$ being shifted log. Absolute risk aversion is decreasing in w. Relative risk aversion is increasing in w if $\zeta < 0$. The shifted power utility function is monotone increasing and bounded on the domain $w \geq 0$ if $\zeta < 0$ and $\rho > 1$.

1.11. Show that condition (ii) in the discussion of second-order stochastic dominance in the end-of-chapter notes implies condition (i); that is, assume $\tilde{y} = \tilde{x} + \tilde{z} + \tilde{\varepsilon}$ where $\tilde{z}$ is a nonpositive random variable and $\mathsf{E}[\tilde{\varepsilon} \mid \tilde{x}+\tilde{z}] = 0$ and show that $\mathsf{E}[u(\tilde{x})] \geq \mathsf{E}[u(\tilde{y})]$ for every monotone concave function u.

Note: The statement of (ii) is that $\tilde{y}$ has the same distribution as $\tilde{x}+\tilde{z}+\tilde{\varepsilon}$, which is a weaker condition than $\tilde{y} = \tilde{x}+\tilde{z}+\tilde{\varepsilon}$, but if $\tilde{y}$ has the same distribution as $\tilde{x}+\tilde{z}+\tilde{\varepsilon}$ and $\tilde{y}' = \tilde{x}+\tilde{z}+\tilde{\varepsilon}$, then $\mathsf{E}[u(\tilde{y})] = \mathsf{E}[u(\tilde{y}')]$ so we can without loss of generality take $\tilde{y} = \tilde{x}+\tilde{z}+\tilde{\varepsilon}$ (though this is not true for the reverse implication (i) $\Rightarrow$ (ii)).

Solution: $\tilde{y}$ equals $\tilde{x}+\tilde{z}$ plus mean-independent noise, so by concavity and Jensen's inequality, $\mathsf{E}[u(\tilde{x}+\tilde{z})] \geq \mathsf{E}[u(\tilde{y})]$, as shown in Section 1.5. Because $\tilde{z}$ is nonpositive and u is monotone, $\mathsf{E}[u(\tilde{x})] \geq \mathsf{E}[u(\tilde{x}+\tilde{z})]$. Therefore, $\mathsf{E}[u(\tilde{x})] \geq \mathsf{E}[u(\tilde{y})]$.

1.12. Show that any monotone LRT utility function is a monotone affine transform of one of the five utility functions: negative exponential, log, power, shifted log, or shifted power. Hint: Consider first the special cases (i) risk tolerance $= A$ and (ii) risk tolerance $= Bw$. In case (i) use the fact that

$$\frac{u''(w)}{u'(w)} = \frac{\mathrm{d}\log u'(w)}{\mathrm{d}w}$$

and in case (ii) use the fact that

$$\frac{wu''(w)}{u'(w)} = \frac{\mathrm{d}\log u'(w)}{\mathrm{d}\log w}$$

to derive formulas for $\log u'(w)$ and hence $u'(w)$ and hence $u(w)$. For the case $A \neq 0$ and $B \neq 0$, define

$$v(w) = u\left(\frac{w-A}{B}\right),$$

show that the risk tolerance of v is Bw, apply the results from case (ii) to v, and then derive the form of u.

Solution: In case (i), set $\alpha = 1/A$. For any constant y,

$$\begin{aligned}
\log u'(w) &= \log u'(y) + \int_y^w \frac{\mathrm{d}\log u'(x)}{\mathrm{d}x}\,\mathrm{d}x \\
&= \log u'(y) + -\alpha \int_y^w \mathrm{d}x \\
&= \log u'(y) - \alpha(w-y)\,.
\end{aligned}$$

Hence,

$$u'(w) = u'(y)\mathrm{e}^{-\alpha(w-y)} = u'(y)\mathrm{e}^{\alpha y}\mathrm{e}^{-\alpha w}\,.$$

This implies

$$\begin{aligned}
u(w) &= u(y) + \int_y^w u'(x)\,\mathrm{d}x \\
&= u(y) + u'(y)\mathrm{e}^{\alpha y}\int_y^w \mathrm{e}^{-\alpha x}\,\mathrm{d}x \\
&= u(y) + u'(y)\mathrm{e}^{\alpha y}\frac{1}{\alpha}\left[\mathrm{e}^{-\alpha y} - \mathrm{e}^{-\alpha w}\right]\,.
\end{aligned}$$

This is an affine transform of $-\mathrm{e}^{-\alpha w}$. For u to be monotone, it must be a monotone affine transform of $-\mathrm{e}^{-\alpha w}$.

In case (ii), set $\rho = 1/B$. For any constant $y > 0$ and any $w > 0$,

$$\begin{aligned}
\log u'(w) &= \log u'(y) + \int_y^w \frac{\mathrm{d}\log u'(x)}{\mathrm{d}\log x}\,\mathrm{d}\log x \\
&= \log u'(y) - \rho \int_y^w \mathrm{d}\log x \\
&= \log u'(y) - \rho(\log w - \log y)\,.
\end{aligned}$$

Hence,

$$u'(w) = u'(y)\mathrm{e}^{-\rho(\log w - \log y)} = u'(y)y^{\rho}w^{-\rho}\,.$$

This implies

$$\begin{aligned}
u(w) &= u(y) + \int_y^w u'(x)\,\mathrm{d}x \\
&= u(y) + u'(y)y^{\rho}\int_y^w x^{-\rho}\,\mathrm{d}x\,.
\end{aligned}$$

If $\rho = 1$, then

$$u(w) = u(y) + u'(y)y(\log w - \log y)\,,$$

which is a monotone affine transform of $\log w$. If $\rho \neq 1$, then

$$u(w) = u(y) + u'(y)y^{\rho}\frac{1}{1-\rho}\left(w^{1-\rho} - y^{1-\rho}\right)\,.$$

which is an affine transform of $w^{1-\rho}/(1-\rho)$. For u to be monotone, it must be a monotone affine transform of $w^{1-\rho}/(1-\rho)$.

For the case $A \neq 0$ and $B \neq 0$, set

$$v(x) = u\left(\frac{x-A}{B}\right)$$

for $x > 0$. This implies

$$\begin{aligned}
-\frac{v'(x)}{v''(x)} &= -B\frac{u'\left(\frac{x-A}{B}\right)}{u''\left(\frac{x-A}{B}\right)} \\
&= B\left[A + B\left(\frac{x-A}{B}\right)\right] \\
&= Bx\,.
\end{aligned}$$

Therefore, from case (ii), on the region $x > 0$, either $v(x) = \log x$ if $B = 1$, or $v(x) = x^{1-\rho}/(1-\rho)$ for $\rho = 1/B$, up to an affine transform. Moreover,

$$u(w) = v(A + Bw)\,.$$

Hence, for w such that $A + Bw > 0$, either $u(w) = \log(A + Bw)$ if $B = 1$, or $u(w) = (A + Bw)^{1-\rho}/(1-\rho)$ for $\rho = 1/B$, up to an affine transform. Setting $\zeta = -A/B$, we have, up to an affine transform, $u(w) = \log(w - \zeta)$ on the region $w > \zeta$ if $B = 1$, or

$$u(w) = \frac{1}{1-\rho}\left(\frac{w-\zeta}{\rho}\right)^{1-\rho},$$

on the region $(w - \zeta)/\rho > 0$. Monotonicity of u in the case $B \neq 1$ requires that u be a monotone affine transform of

$$\frac{\rho}{1-\rho}\left(\frac{w-\zeta}{\rho}\right)^{1-\rho}.$$

1.13. Show that risk neutrality [$u(w) = w$ for all w] can be regarded as a limiting case of negative exponential utility as $\alpha \to 0$ by showing that there are monotone affine transforms of negative exponential utility that converges to w as $\alpha \to 0$. Hint: Take an exact first-order Taylor series expansion of negative exponential utility, expanding in α around $\alpha = 0$. Writing the expansion as $c_0 + c_1\alpha$, show that

$$\frac{-\mathrm{e}^{-\alpha w} - c_0}{\alpha} \to w$$

as $\alpha \to 0$.

Solution: Set $f(\alpha) = -\mathrm{e}^{-\alpha w}$. We have

$$f(\alpha) = f(0) + f'(\hat{\alpha})\alpha$$

for some $0 < \hat{\alpha} < \alpha$, and $f'(\alpha) = w\mathrm{e}^{-\alpha w}$. Thus,

$$-\mathrm{e}^{-\alpha w} = -1 + w\mathrm{e}^{-\hat{\alpha} w}\alpha\,.$$

This implies

$$\frac{-\mathrm{e}^{-\alpha w} + 1}{\alpha} = w\mathrm{e}^{-\hat{\alpha} w} \to w\,,$$

as $\alpha \to 0$.