Sample Midterm Exam Solutions

BUSI 521 / ECON 505 — Asset Pricing

1. Assume there are three equally likely states. Two assets with prices $p_1 = p_2 = 1$ and payoffs $(1, 2, 1)$ and $(0, 1, 3)$.

   (a) **State price vectors.** Let $q = (q_1, q_2, q_3)$ be a state price vector. The pricing equations are:
   $$q_1 + 2q_2 + q_3 = 1\,, \qquad 0 \cdot q_1 + q_2 + 3q_3 = 1\,.$$
   From the second equation, $q_2 = 1 - 3q_3$. Substituting into the first:
   $$q_1 + 2(1 - 3q_3) + q_3 = 1 \implies q_1 = -1 + 5q_3\,.$$
   Thus the one-dimensional family is $q = (-1 + 5q_3,\ 1 - 3q_3,\ q_3)$ for $q_3 > 0$ with $q_1 > 0$ and $q_2 > 0$, i.e., $1/5 < q_3 < 1/3$.

   (b) **SDF spanned by the assets.** Each state has probability $1/3$, so the SDF is $\widetilde{m} = q_i/(1/3) = 3q_i$ in state i. Let X be the 2×3 payoff matrix:
   $$X = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$
   The SDF spanned by the assets is the projection of any SDF onto the span of the asset payoffs. We need $\widetilde{m} = X'\theta$ for some $\theta \in \mathbb{R}^2$, and the pricing conditions $\frac{1}{3}X\,\widetilde{m} = p$ give the system
   $$\frac{1}{3}XX'\theta = p = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
   We compute $\frac{1}{3}XX' = \frac{1}{3}\begin{pmatrix} 6 & 5 \\ 5 & 10 \end{pmatrix}$, so
   $$\theta = 3\begin{pmatrix} 6 & 5 \\ 5 & 10 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \cdot \frac{1}{35}\begin{pmatrix} 10 & -5 \\ -5 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{3}{35}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/7 \\ 3/35 \end{pmatrix}.$$
   Hence $\widetilde{m} = X'\theta$:
   $$\widetilde{m} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3/7 \\ 3/35 \end{pmatrix} = \begin{pmatrix} 3/7 \\ 6/7 + 3/35 \\ 3/7 + 9/35 \end{pmatrix} = \begin{pmatrix} 15/35 \\ 33/35 \\ 24/35 \end{pmatrix}.$$

2. **MV frontier FOC and factor pricing.**

   Assume there is a risk-free asset with return R_f. Let $\tilde{\mathbf{R}}$ be the vector of risky asset returns with mean μ and covariance matrix Σ. A portfolio on the mean-variance frontier solves
   $$\min_{\pi}\ \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad \pi'(\mu - R_f\iota) = \mu_{\text{targ}} - R_f\,.$$
   The first-order condition (with Lagrange multiplier δ) is
   $$\Sigma\pi = \delta(\mu - R_f\iota) \implies \pi = \delta\,\Sigma^{-1}(\mu - R_f\iota)\,.$$
   The return on this portfolio is $\widetilde{R}_p = R_f + \pi'(\tilde{\mathbf{R}} - R_f\iota)$. For any return $\widetilde{R}_i$,
   $$\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p) = \pi'\Sigma e_i = \delta(\mu - R_f\iota)'\Sigma^{-1}\Sigma e_i = \delta(\mu_i - R_f)\,,$$

where e_i selects asset i. Hence $\mu_i - R_f = \frac{1}{\delta}\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p)$, which is a factor model with $\widetilde{R}_p$ as the single factor:

$$\mathsf{E}[\widetilde{R}_i] - R_f = \lambda \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p)\,, \quad \lambda = \frac{1}{\delta}\,.$$

3. **CRRA risk-free rate with lognormal consumption.**

   Assume $\log \tilde{c}_1 \sim N(\log c_0 + \mu,\ \sigma^2)$. The SDF is $\widetilde{m} = \delta(\tilde{c}_1/c_0)^{-\rho}$, so

   $$\frac{1}{R_f} = \mathsf{E}[\widetilde{m}] = \delta\,\mathsf{E}\Big[e^{-\rho\log(\tilde{c}_1/c_0)}\Big] = \delta\, e^{-\rho\mu + \rho^2\sigma^2/2}\,,$$

   using the moment generating function of a normal. Thus,

   $$\log R_f = -\log\delta + \rho\mu - \frac{\rho^2\sigma^2}{2}\,.$$

   (a) This is the formula derived above.

   (b) When μ is larger, expected consumption growth is higher. Investors want to borrow against future income to smooth consumption, pushing up R_f. When σ is smaller, there is less precautionary saving motive, so investors save less, and R_f rises.

   (c) The effect of ρ is ambiguous because ρ appears in both the $+\rho\mu$ term (higher ρ increases R_f, the consumption-smoothing effect) and the $-\rho^2\sigma^2/2$ term (higher ρ decreases R_f, the precautionary-savings effect).

4. **Linear risk tolerance, sharing rules, Pareto optimality, two-fund separation.**

   All investors have linear risk tolerance $\tau_h(w) = A_h + Bw$ with the same cautiousness parameter B.

   (a) **Affine sharing rules.** In a Pareto optimum, the first-order conditions $\lambda_h u_h'(\tilde{w}_h) = \tilde{\eta}$ yield each investor's consumption as a function of the social marginal utility $\tilde{\eta}$. Because risk tolerance is affine with the same slope B, Pareto optimal sharing rules are affine:

   $$\tilde{w}_h = a_h + b_h\tilde{w}_m\,, \quad \textstyle\sum_h a_h = 0,\ \sum_h b_h = 1\,.$$

   (b) **Pareto optimality with a risk-free asset.** Because sharing rules are affine, each investor's optimal consumption $\tilde{w}_h = a_h + b_h\tilde{w}_m$ can be replicated by holding a_h/R_f units of the risk-free asset and fraction b_h of the market portfolio. No other securities are needed, so as long as a risk-free asset exists, the competitive equilibrium allocation is Pareto optimal even without complete markets.

   (c) **Two-fund separation.** Since every investor holds only the risk-free asset and the market portfolio (a consequence of affine sharing rules), there is two-fund separation: the risk-free asset and the tangency (market) portfolio are the two funds.

5. **Dynamic programming with CRRA utility.**

   At date $T-1$, the investor chooses consumption c and portfolio π to maximize

   $$\frac{1}{1-\rho}c^{1-\rho} + \delta\,\mathsf{E}\left[\frac{1}{1-\rho}\big((w-c)\pi'\tilde{R}_T\big)^{1-\rho}\right].$$

   Because u is CRRA, we can write the second term as

   $$\delta\,\frac{(w-c)^{1-\rho}}{1-\rho}\,\mathsf{E}\Big[(\pi'\tilde{R}_T)^{1-\rho}\Big]\,.$$

The optimal portfolio π^* maximizes $\mathsf{E}\Big[\frac{1}{1-\rho}(\pi'\tilde{R}_T)^{1-\rho}\Big]$, i.e., it achieves the maximum B. Hence the value function becomes

$$J_{T-1}(w) = \max_c \left\{ \frac{c^{1-\rho}}{1-\rho} + \delta B (w-c)^{1-\rho} \right\}.$$

The FOC is $c^{-\rho} = \delta B(1-\rho)(w-c)^{-\rho}$, giving $c = \alpha w$ for a constant α. Substituting back:

$$J_{T-1}(w) = \frac{1}{1-\rho}\left[\alpha^{1-\rho} + \delta B(1-\rho)(1-\alpha)^{1-\rho}\right] w^{1-\rho} = \frac{1}{1-\rho} A_{T-1}\, w^{1-\rho},$$

where $A_{T-1} = \alpha^{1-\rho} + \delta B(1-\rho)(1-\alpha)^{1-\rho}$ is a constant.

6. **SDF formula with R_f.**

   (a) Define $\widetilde{m} = \frac{1}{R_f} + \left(\iota - \frac{1}{R_f}\mu\right)' \Sigma^{-1}(\tilde{\mathbf{R}} - \mu)$. We verify $\mathsf{E}[\widetilde{m}\widetilde{R}_i] = 1$ for each risky asset i and $\mathsf{E}[\widetilde{m}]R_f = 1$.
   $\mathsf{E}[\widetilde{m}] = \frac{1}{R_f}$ since $\mathsf{E}[\tilde{\mathbf{R}} - \mu] = 0$. Thus $\mathsf{E}[\widetilde{m}]R_f = 1$.

   For each risky return $\widetilde{R}_i$:

   $$\begin{aligned}\mathsf{E}[\widetilde{m}\widetilde{R}_i] &= \frac{\mu_i}{R_f} + \left(\iota - \frac{\mu}{R_f}\right)' \Sigma^{-1}\operatorname{cov}(\tilde{\mathbf{R}}, \widetilde{R}_i) \\ &= \frac{\mu_i}{R_f} + \left(\iota - \frac{\mu}{R_f}\right)' \Sigma^{-1}\Sigma e_i = \frac{\mu_i}{R_f} + 1 - \frac{\mu_i}{R_f} = 1.\end{aligned}$$

   (b) The SDF $\widetilde{m}$ is in the span of a constant and the returns $\tilde{\mathbf{R}}$, so it is spanned by the assets. Its cost is $\mathsf{E}[\widetilde{m} \cdot \widetilde{m}/\text{cost}]$... more directly: the payoff $\widetilde{m}$ costs $\mathsf{E}[\widetilde{m}^2]$ (using $\widetilde{m}$ itself as the SDF to price $\widetilde{m}$). Dividing $\widetilde{m}$ by its cost gives a return. The frontier return proportional to an SDF is

   $$\widetilde{R}_p = \frac{\widetilde{m}}{\mathsf{E}[\widetilde{m}^2]}.$$

   Since $\widetilde{m}$ is an affine function of $\tilde{\mathbf{R}}$, $\widetilde{R}_p$ is a return on the mean-variance frontier. Thus $\widetilde{m}$ is proportional to a frontier return $\pi'\tilde{\mathbf{R}} + (1 - \iota'\pi)R_f$.

7. **CRRA risk-free return with lognormal consumption.**
   Let $\Delta \log c = \log(\tilde{c}_1/c_0) \sim N(\mu, \sigma^2)$. The SDF is $\widetilde{m} = \delta\, e^{-\rho\,\Delta\log c}$.

   (a)
   $$\frac{1}{R_f} = \mathsf{E}[\widetilde{m}] = \delta\, e^{-\rho\mu + \rho^2\sigma^2/2} \quad\Longrightarrow\quad R_f = \frac{1}{\delta}\, e^{\rho\mu - \rho^2\sigma^2/2}.$$

   (b) Higher μ means higher expected consumption growth; investors want to borrow against future consumption to smooth, raising R_f. Higher σ increases precautionary saving demand, lowering R_f.

8. **CARA optimal portfolio with normal returns.**
   Let $u(w) = -e^{-\alpha w}$. With a risk-free asset and risky returns $\tilde{\mathbf{R}} \sim N(\mu, \Sigma)$, end-of-period wealth is $\tilde{w} = w_0 R_f + \phi'(\tilde{\mathbf{R}} - R_f\iota)$. This is normal with mean $w_0R_f + \phi'(\mu - R_f\iota)$ and variance $\phi'\Sigma\phi$. Expected utility is

   $$-\exp\!\left(-\alpha\left[w_0R_f + \phi'(\mu - R_f\iota)\right] + \frac{\alpha^2}{2}\phi'\Sigma\phi\right).$$

Maximizing over ϕ: the FOC is $\mu - R_f\iota - \alpha\Sigma\phi = 0$, giving the optimal portfolio

$$\phi^* = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f\iota)\,.$$

9. **Complete market, unique state price vector.**

   There are n assets and k states, with $n \geq k$ for completeness. The payoff matrix X is $n \times k$ with $\text{rank}(X) = k$. The pricing condition is $p = Xq$, where $q \in \mathbb{R}^k$ is the state price vector and p is the price vector. Since X has full column rank, $X'X$ is invertible, and the unique state price vector is

   $$q = (X'X)^{-1}X'p\,.$$

   This is the unique solution because X has rank k (the market is complete), so the system $Xq = p$ has exactly one solution.

10. **SDF formula with R_f (projection interpretation).**

    (a) Same as Question 6(a). The random variable

    $$\widetilde{m}^* = \frac{1}{R_f} + \left(\iota - \frac{1}{R_f}\mu\right)' \Sigma^{-1}(\tilde{\mathbf{R}} - \mu)$$

    is an SDF, verified by checking $\mathsf{E}[\widetilde{m}^*\widetilde{R}_i] = 1$ for all risky returns and $\mathsf{E}[\widetilde{m}^*]R_f = 1$.

    (b) Let $\widetilde{m}$ be any SDF. Decompose $\widetilde{m} = \widetilde{m}_p + \varepsilon$, where $\widetilde{m}_p$ is the projection of $\widetilde{m}$ onto the span of a constant and the returns, and ε is orthogonal to all returns. Then $\widetilde{m}_p$ is the unique SDF in the span of the returns. Because $\widetilde{m}^*$ is also in the span of the returns and is an SDF, by uniqueness of the Riesz representation, $\widetilde{m}^* = \widetilde{m}_p$. Thus the random variable in part (a) equals the orthogonal projection of any SDF onto the span of the returns.

11. **Tangency portfolio.**

    With a risk-free asset, the tangency portfolio maximizes the Sharpe ratio:

    $$\max_{\pi} \frac{\pi'(\mu - R_f\iota)}{\sqrt{\pi'\Sigma\pi}}\,.$$

    The solution is $\pi_{\text{tang}} = c\,\Sigma^{-1}(\mu - R_f\iota)$ for a normalizing constant $c = 1/[\iota'\Sigma^{-1}(\mu - R_f\iota)]$ so that $\iota'\pi_{\text{tang}} = 1$.

    The tangency portfolio exists provided $\iota'\Sigma^{-1}(\mu - R_f\iota) \neq 0$, i.e., $R_f \neq B/C$ where $B = \iota'\Sigma^{-1}\mu$ and $C = \iota'\Sigma^{-1}\iota$. If $R_f = B/C$ (the expected return of the GMV portfolio), no tangency portfolio exists.

12. **Factor pricing from SDF.**

    Suppose $\widetilde{m} = a + b'\tilde{X}$ is an SDF. For any return $\widetilde{R}$:

    $$1 = \mathsf{E}[\widetilde{m}\widetilde{R}] = a\,\mathsf{E}[\widetilde{R}] + b'\mathsf{E}[\tilde{X}\widetilde{R}] = a\,\mathsf{E}[\widetilde{R}] + b'\left(\text{cov}(\tilde{X}, \widetilde{R}) + \mathsf{E}[\tilde{X}]\mathsf{E}[\widetilde{R}]\right).$$

    Hence,

    $$\mathsf{E}[\widetilde{R}]\left(a + b'\mathsf{E}[\tilde{X}]\right) = 1 - b'\,\text{cov}(\tilde{X}, \widetilde{R})\,.$$

    Applying this to $\widetilde{R} = R_f$ (if a risk-free asset exists) gives $a + b'\mathsf{E}[\tilde{X}] = 1/R_f$, so

    $$\mathsf{E}[\widetilde{R}] - R_f = -R_f\,b'\,\text{cov}(\tilde{X}, \widetilde{R})\,.$$

This is a factor pricing model with factors $\tilde{X}$ and risk premium vector $\lambda = -R_f\, b$.

If there is no risk-free asset, define $R_z = 1/(a+b'\mathsf{E}[\tilde{X}])$ and the same algebra yields $\mathsf{E}[\widetilde{R}]-R_z = -R_z\, b'\operatorname{cov}(\tilde{X}, \widetilde{R})$.

13. **Generalized CAPM with CRRA.**

    With a representative investor with CRRA ρ and a risk-free asset, $\widetilde{m} = \delta(\tilde{c}_1/c_0)^{-\rho}$ is an SDF. In equilibrium, $\tilde{c}_1 \propto \tilde{R}_m$, so $\widetilde{m} \propto \tilde{R}_m^{-\rho}$. For each asset i:

    $$\mathsf{E}[\widetilde{R}_i] - R_f = -\frac{\operatorname{cov}(\tilde{R}_m^{-\rho}, \widetilde{R}_i)}{\mathsf{E}[\tilde{R}_m^{-\rho}]} \cdot R_f = \frac{\operatorname{cov}(\widetilde{R}_i, \tilde{R}_m^{-\rho})}{\operatorname{cov}(\tilde{R}_m, \tilde{R}_m^{-\rho})} \Big(\mathsf{E}[\tilde{R}_m] - R_f\Big)\,.$$

    Define the generalized beta:

    $$\gamma_i = \frac{\operatorname{cov}(\widetilde{R}_i, \tilde{R}_m^{-\rho})}{\operatorname{cov}(\tilde{R}_m, \tilde{R}_m^{-\rho})}\,.$$

    Then $\mathsf{E}[\widetilde{R}_i] - R_f = \gamma_i(\mathsf{E}[\tilde{R}_m] - R_f)$.

    When $\rho = 1$, $\tilde{R}_m^{-\rho} = 1/\tilde{R}_m$, and γ_i reduces to something close to the standard CAPM beta. When $\rho \neq 1$, this is a nonlinear generalization.

14. **Log utility Pareto optimal sharing.**

    With log utility $u_h(w) = \log w$, the Pareto optimality FOC is $\lambda_h/\tilde{w}_h = \tilde{\eta}$ for all h, where $\tilde{\eta}$ is the social marginal utility. Thus $\tilde{w}_h = \lambda_h/\tilde{\eta}$. Summing over h:

    $$\tilde{w}_m = \sum_h \tilde{w}_h = \frac{1}{\tilde{\eta}} \sum_h \lambda_h\,.$$

    Hence $\tilde{\eta} = (\sum_h \lambda_h)/\tilde{w}_m$ and

    $$\tilde{w}_h = \frac{\lambda_h}{\sum_j \lambda_j}\, \tilde{w}_m = b_h\, \tilde{w}_m\,,$$

    where $b_h = \lambda_h/\sum_j \lambda_j$. This is affine (in fact, proportional) in $\tilde{w}_m$.

15. **Verify SDF.**

    Four equally probable states, $R_f = 1.5$, two risky assets with payoffs and prices as given. The proposed SDF is $\widetilde{m} = (1/3,\ 1/3,\ 1,\ 1)$. We verify:

    Risk-free asset: $\mathsf{E}[\widetilde{m}]R_f = \frac{1}{4}(1/3+1/3+1+1)\cdot 1.5 = \frac{1}{4}\cdot\frac{8}{3}\cdot 1.5 = 1.$ ✓

    Asset 1 (price $p_1 = 1$, payoffs $(3,3,1,1)$):

    $$\mathsf{E}[\widetilde{m}\tilde{x}_1] = \frac{1}{4}\left(\frac{1}{3}\cdot 3 + \frac{1}{3}\cdot 3 + 1\cdot 1 + 1\cdot 1\right) = \frac{1}{4}(1+1+1+1) = 1 = p_1\,. \checkmark$$

    Asset 2 (price $p_2 = 2$, payoffs $(1,5,2,4)$):

    $$\mathsf{E}[\widetilde{m}\tilde{x}_2] = \frac{1}{4}\left(\frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 5 + 1\cdot 2 + 1\cdot 4\right) = \frac{1}{4}\left(\frac{1}{3} + \frac{5}{3} + 2 + 4\right) = \frac{1}{4}\cdot 8 = 2 = p_2\,. \checkmark$$

16. **Complete market test.**

    With two investors and three states, Pareto optimality implies that each investor's consumption is a function of aggregate consumption. Aggregate wealth by state: $(3,7,8)$. Investor 1's wealth $(1,4,3)$ is *not* a monotone function of aggregate wealth (states ordered by aggregate wealth give $(3, 7, 8) \to (1, 4, 3)$, but $3 < 4$ while $8 > 7$, and $3 < 4$ okay, but the ordering $1, 4, 3$ is not monotone with $3, 7, 8$).

Specifically, investor 1's wealth is 1 when aggregate is 3, 4 when aggregate is 7, and 3 when aggregate is 8. Since investor 1 is risk averse, u_1' is decreasing, so the sharing rule $\tilde{w}_1 = f(\tilde{w}_m)$ must be increasing if the allocation is Pareto optimal. But $f(7) = 4 > 3 = f(8)$, violating monotonicity. Therefore the allocation is *not* Pareto optimal, which means the market is *not* complete (since competitive equilibria in complete markets are Pareto optimal).

17. **Risky asset demand $\phi'(w)$.**

    In a single-period model with a risk-free asset and one risky asset with excess return $\tilde{R} - R_f$, the investor maximizes $\mathsf{E}[u(wR_f + \phi(\tilde{R} - R_f))]$. The FOC is

    $$\mathsf{E}\Big[u'\Big(wR_f + \phi(\tilde{R} - R_f)\Big)\,(\tilde{R} - R_f)\Big] = 0\,.$$

    Differentiating implicitly with respect to w:

    $$\mathsf{E}\Big[u''(\tilde{w})\,\Big(R_f + \phi'(w)(\tilde{R} - R_f)\Big)\,(\tilde{R} - R_f)\Big] = 0\,.$$

    Solving for $\phi'(w)$:

    $$\phi'(w) = -\frac{R_f\,\mathsf{E}[u''(\tilde{w})(\tilde{R} - R_f)]}{\mathsf{E}[u''(\tilde{w})(\tilde{R} - R_f)^2]}\,.$$

    The denominator is negative (since $u'' < 0$). The sign of $\phi'(w)$ equals the sign of $\mathsf{E}[u''(\tilde{w})(\tilde{R} - R_f)]$. This is positive (meaning $\phi'(w) > 0$, investment in the risky asset increases with wealth) when the investor has decreasing absolute risk aversion (DARA).

18. **Two-fund separation.**

    Two-fund separation means that all investors hold portfolios that are combinations of the same two mutual funds (typically the risk-free asset and a single portfolio of risky assets).

    The assumption that implies two-fund separation is that all investors have **linear risk tolerance with the same cautiousness parameter** (the same slope B in $\tau(w) = A_h + Bw$). This includes CARA (all same $B = 0$), CRRA (all same ρ, so $B = 1/\rho$), and quadratic utility.

    Under this assumption, sharing rules are affine, so each investor holds a linear combination of the risk-free asset and the market portfolio.

19. **Risk-neutral probability.**

    Define $Q(A) = R_f\,\mathsf{E}[\widetilde{m}\mathbf{1}_A]$ for each event A. For any return $\widetilde{R}$:

    $$\mathsf{E}^*[\widetilde{R}] = R_f\,\mathsf{E}[\widetilde{m}\widetilde{R}]\,.$$

    Since $\widetilde{m}$ is an SDF, $\mathsf{E}[\widetilde{m}\widetilde{R}] = 1$ for every return. Therefore,

    $$\mathsf{E}^*[\widetilde{R}] = R_f \cdot 1 = R_f\,.$$

20. **GMV portfolio.**

    The global minimum variance (GMV) portfolio minimizes $\pi'\Sigma\pi$ subject to $\iota'\pi = 1$. Using a Lagrange multiplier:

    $$\mathcal{L} = \frac{1}{2}\pi'\Sigma\pi - \lambda(\iota'\pi - 1)\,.$$

FOC: $\Sigma\pi = \lambda\iota$, so $\pi = \lambda\Sigma^{-1}\iota$. Imposing $\iota'\pi = 1$:

$$\lambda\,\iota'\Sigma^{-1}\iota = 1 \implies \lambda = \frac{1}{\iota'\Sigma^{-1}\iota}\,.$$

Thus,

$$\pi_{\text{gmv}} = \frac{\Sigma^{-1}\iota}{\iota'\Sigma^{-1}\iota}\,.$$

21. **SDF as return (two properties).**

    If $\widetilde{m}$ is an SDF and $a + b\widetilde{m}$ is a return, then:

    (i) **It is on the mean-variance frontier.** Any return that is an affine function of an SDF is on the mean-variance frontier, because the frontier return $\tilde{R}_p = \widetilde{m}_p/\mathsf{E}[\widetilde{m}_p^2]$ is proportional to the projected SDF.

    (ii) **It is on the inefficient part of the frontier** (assuming $b > 0$). The SDF is a decreasing function of wealth in states where wealth is high (good states have low SDF values). A return positively related to $\widetilde{m}$ is high in bad states and low in good states, meaning it has a low expected return for its level of variance—it is inefficient. (Specifically, $\tilde{R}_p$ is the minimum second-moment return, which is on the inefficient part.)

22. Same as Question 2.

    A return $\widetilde{R}_p$ is on the mean-variance frontier if and only if $\widetilde{R}_p = R_f + \delta(\mu - R_f\iota)'\Sigma^{-1}(\tilde{\mathbf{R}} - \mu)$ for some δ. The FOC for frontier membership is $\Sigma\pi = \delta(\mu - R_f\iota)$, i.e., $\pi = \delta\Sigma^{-1}(\mu - R_f\iota)$.

    This relates to factor pricing because: for any return $\widetilde{R}_i$,

    $$\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p) = e_i'\Sigma\pi = \delta(\mu_i - R_f)\,,$$

    so

    $$\mathsf{E}[\widetilde{R}_i] - R_f = \frac{1}{\delta}\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p) = \frac{\mathsf{E}[\widetilde{R}_p] - R_f}{\operatorname{var}(\widetilde{R}_p)}\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p)\,.$$

    This is a single-factor model with $\widetilde{R}_p$ as the factor and the price of risk equal to the Sharpe ratio squared divided by the risk premium, or equivalently $\beta_i(\mathsf{E}[\widetilde{R}_p] - R_f)$ where $\beta_i = \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p)/\operatorname{var}(\widetilde{R}_p)$.

23. **CAPM SDF.**

    If the CAPM holds, the market return $\tilde{R}_m$ is on the mean-variance frontier. From the SDF representation of frontier returns, $\widetilde{m} = a - b\tilde{R}_m$ is an SDF for some a, b.

    We need $\mathsf{E}[\widetilde{m}] = 1/R_f$ and $\mathsf{E}[\widetilde{m}\tilde{R}_m] = 1$. These give:

    $$\begin{aligned} a - b\,\mathsf{E}[\tilde{R}_m] &= \frac{1}{R_f}\,, \\ a\,\mathsf{E}[\tilde{R}_m] - b\,\mathsf{E}[\tilde{R}_m^2] &= 1\,. \end{aligned}$$

    From the first equation, $a = 1/R_f + b\,\mathsf{E}[\tilde{R}_m]$. Substituting into the second:

    $$\begin{aligned} \left(\frac{1}{R_f} + b\,\mathsf{E}[\tilde{R}_m]\right)\mathsf{E}[\tilde{R}_m] - b\,\mathsf{E}[\tilde{R}_m^2] &= 1 \\ \frac{\mathsf{E}[\tilde{R}_m]}{R_f} - b\,\operatorname{var}(\tilde{R}_m) &= 1\,. \end{aligned}$$

Hence,

$$b = \frac{\mathsf{E}[\tilde{R}_m]/R_f - 1}{\operatorname{var}(\tilde{R}_m)} = \frac{\mathsf{E}[\tilde{R}_m] - R_f}{R_f \ \operatorname{var}(\tilde{R}_m)}\,,$$

and

$$a = \frac{1}{R_f} + \frac{\mathsf{E}[\tilde{R}_m] - R_f}{R_f \ \operatorname{var}(\tilde{R}_m)}\, \mathsf{E}[\tilde{R}_m] = \frac{1}{R_f}\left(1 + \frac{\mathsf{E}[\tilde{R}_m](\mathsf{E}[\tilde{R}_m] - R_f)}{\operatorname{var}(\tilde{R}_m)}\right).$$

24. **Factor risk premium = expected excess return.**

    Suppose $\tilde{f}$ is an excess return that is a pricing factor: $\mathsf{E}[\widetilde{R}_i] - R_f = \beta_i \lambda$ for $\beta_i = \operatorname{cov}(\widetilde{R}_i, \tilde{f}) / \operatorname{var}(\tilde{f})$. Apply this to the return $R_f + \tilde{f}$ (which is a return since $\tilde{f}$ is an excess return):

    $$\mathsf{E}[R_f + \tilde{f}] - R_f = \beta_f\, \lambda\,.$$

    Here $\beta_f = \operatorname{cov}(R_f + \tilde{f}, \tilde{f}) / \operatorname{var}(\tilde{f}) = 1$. Thus $\mathsf{E}[\tilde{f}] = \lambda$. The factor risk premium equals the expected excess return of the factor.

25. **Macro factors to return factors.**

    If there is a factor pricing model $\mathsf{E}[\widetilde{R}_i] - R_f = \beta_i' \lambda$ with macroeconomic factors $\tilde{F}$, then define the *factor-mimicking portfolios*: for each factor $\tilde{f}_j$, project it onto the space of returns to obtain a return $\tilde{R}_j$ that has maximum correlation with $\tilde{f}_j$. Specifically, the mimicking return for factor j is

    $$\tilde{R}_j = R_f + \operatorname{cov}(\tilde{f}_j, \tilde{\mathbf{R}})' \Sigma^{-1} (\tilde{\mathbf{R}} - R_f \iota) \cdot c_j$$

    for an appropriate constant c_j. Because the covariance of any return with $\tilde{f}_j$ is proportional to its covariance with $\tilde{R}_j$, there is a factor pricing model with the returns $\tilde{R}_1, \ldots, \tilde{R}_k$ as factors. These return factors are the projections of the macro factors onto the return space.

26. **CRRA risk premium formula.**

    With CRRA ρ and a risk-free asset, $\widetilde{m} = \gamma \tilde{R}_m^{-\rho}$ is an SDF for some constant γ. For any return $\widetilde{R}$:

    $$\mathsf{E}[\widetilde{R}] - R_f = -R_f \ \operatorname{cov}(\widetilde{m}, \widetilde{R}) / \mathsf{E}[\widetilde{m}] \cdot \mathsf{E}[\widetilde{m}] = -\frac{\operatorname{cov}(\widetilde{m}, \widetilde{R})}{\mathsf{E}[\widetilde{m}]} \cdot R_f\,.$$

    More directly, $\mathsf{E}[\widetilde{R}] - R_f = -R_f \ \operatorname{cov}(\gamma \tilde{R}_m^{-\rho}, \widetilde{R}) \cdot R_f \ldots$ Let us use the standard derivation. From $\mathsf{E}[\widetilde{m}\widetilde{R}] = 1$ and $\mathsf{E}[\widetilde{m}] = 1/R_f$:

    $$\mathsf{E}[\widetilde{R}] - R_f = -\frac{\operatorname{cov}(\widetilde{m}, \widetilde{R})}{\mathsf{E}[\widetilde{m}]}\,.$$

    Since $\widetilde{m} \propto \tilde{R}_m^{-\rho}$:

    $$\mathsf{E}[\widetilde{R}] - R_f = -\frac{\operatorname{cov}(\tilde{R}_m^{-\rho}, \widetilde{R})}{\mathsf{E}[\tilde{R}_m^{-\rho}]}\,.$$

    Applying this to $\widetilde{R} = \tilde{R}_m$:

    $$\mathsf{E}[\tilde{R}_m] - R_f = -\frac{\operatorname{cov}(\tilde{R}_m^{-\rho}, \tilde{R}_m)}{\mathsf{E}[\tilde{R}_m^{-\rho}]}\,.$$

    Dividing:

    $$\mathsf{E}[\widetilde{R}] - R_f = \frac{\mathsf{E}[\tilde{R}_m] - R_f}{\operatorname{cov}(\tilde{R}_m, \tilde{R}_m^{-\rho})} \ \operatorname{cov}(\widetilde{R}, \tilde{R}_m^{-\rho})\,.$$

27. **Equity premium with lognormal consumption.**

    With CRRA ρ and $\log(\tilde{c}_1/c_0) \sim N(\mu, \sigma^2)$, we showed in Question 3 that $\log R_f = -\log\delta + \rho\mu - \rho^2\sigma^2/2$.

    For the market return, in equilibrium $\tilde{R}_m \propto \tilde{c}_1/c_0$, so $\log\tilde{R}_m = \log(\tilde{c}_1/c_0) + \text{const}$. Using the Euler equation $1 = \mathsf{E}[\widetilde{m}\tilde{R}_m] = \delta\,\mathsf{E}[(\tilde{c}_1/c_0)^{1-\rho}]$:

    $$\log\mathsf{E}[\tilde{R}_m] = -\log\delta + (1-\rho)\left(-\mu + \frac{(1-\rho)\sigma^2}{2}\right)^{-1}\cdots$$

    More carefully: let $z = \log(\tilde{c}_1/c_0) \sim N(\mu, \sigma^2)$.

    $$\log\mathsf{E}[\tilde{R}_m] = \mu + \sigma^2/2 + \text{const terms that cancel}\,.$$

    The equity premium is:

    $$\log\mathsf{E}[\tilde{R}_m] - \log R_f = \rho\sigma^2\,.$$

    Proof: $\mathsf{E}[\widetilde{m}\tilde{R}_m] = 1$ with $\widetilde{m} = \delta e^{-\rho z}$ and $\tilde{R}_m \propto e^z$ gives $\log\mathsf{E}[\tilde{R}_m] = -\log\delta + (1-\rho)\mu + (1-\rho)^2\sigma^2/2 + \sigma^2/2$... Actually, a clean derivation uses:

    $$\log\mathsf{E}[\tilde{R}_m] - \log R_f = -\operatorname{cov}(\log\widetilde{m}, \log\tilde{R}_m) - \frac{1}{2}\operatorname{var}(\log\tilde{R}_m)\,.$$

    But the simplest route: $\mathsf{E}[\widetilde{m}\tilde{R}_m] = 1$ implies $\operatorname{cov}(\widetilde{m}, \tilde{R}_m) = -\mathsf{E}[\widetilde{m}]\mathsf{E}[\tilde{R}_m] + 1$, so

    $$\mathsf{E}[\tilde{R}_m] - R_f = -R_f\operatorname{cov}(\widetilde{m}, \tilde{R}_m)\,.$$

    With $\widetilde{m} = \delta e^{-\rho z}$, $\tilde{R}_m = \nu^{-1}e^z$ (where $\nu = \delta\mathsf{E}[e^{(1-\rho)z}]$), and using lognormal properties:

    $$\log\mathsf{E}[\tilde{R}_m] - \log R_f = \rho\sigma^2\,.$$

28. **Equity premium puzzle.**

    The equity premium puzzle (Mehra and Prescott, 1985) is the observation that the historically observed equity premium (about 6% per year) is too large to be explained by standard consumption-based asset pricing models with reasonable levels of risk aversion.

    From Question 27, $\log\mathsf{E}[\tilde{R}_m] - \log R_f = \rho\sigma^2$. With $\sigma \approx 0.035$ (consumption growth std. dev.), we get $\rho\sigma^2 \approx \rho \times 0.00125$. To match a 6% premium requires $\rho \approx 48$, which is an implausibly high level of risk aversion. Additionally, with ρ that high, the risk-free rate formula predicts an extremely high R_f (the risk-free rate puzzle).

29. **Market return with IID consumption growth.**

    With CRRA ρ, discount factor δ, and IID consumption growth C_{t+1}/C_t, the market portfolio is a claim to the consumption stream. In the infinite-horizon case, the price of the market at date t is

    $$P_t = \mathsf{E}_t\left[\sum_{s=1}^{\infty}\delta^s\left(\frac{C_{t+s}}{C_t}\right)^{-\rho}C_{t+s}\right] = C_t\sum_{s=1}^{\infty}\delta^s\,\mathsf{E}\left[\left(\frac{C_{t+s}}{C_t}\right)^{1-\rho}\right].$$

    By the IID assumption, $\mathsf{E}[(C_{t+s}/C_t)^{1-\rho}] = \left(\mathsf{E}[(C_{t+1}/C_t)^{1-\rho}]\right)^s$. Define $\nu = \delta\,\mathsf{E}[(C_{t+1}/C_t)^{1-\rho}]$. Then $P_t = C_t\cdot\nu/(1-\nu)$. The market return is

    $$R_{m,t+1} = \frac{P_{t+1} + C_{t+1}}{P_t} = \frac{C_{t+1}[\nu/(1-\nu) + 1]}{C_t\cdot\nu/(1-\nu)} = \frac{1}{\nu}\cdot\frac{C_{t+1}}{C_t}\,.$$

30. **Bellman equation with CRRA and state variable.**

The Bellman equation is

$$J(w,x) = \max_{c,\pi} \left\{ \frac{c^{1-\rho}}{1-\rho} + \delta\, \mathsf{E}\Big[J\Big((w-c)\pi' \tilde{R}(x),\, X' \Big) \;\Big|\; X = x \Big] \right\}.$$

Guess $J(w,x) = \frac{1}{1-\rho} w^{1-\rho} f(x)$. With CRRA, the optimal $c = \alpha(x)\, w$ for some function α. Substituting:

$$\begin{aligned} \frac{w^{1-\rho} f(x)}{1-\rho} &= \frac{(\alpha w)^{1-\rho}}{1-\rho} + \delta\, \mathsf{E}\left[\frac{((1-\alpha)w)^{1-\rho} (\pi' \tilde{R})^{1-\rho} f(X')}{1-\rho} \;\Big|\; x \right] \\ &= \frac{w^{1-\rho}}{1-\rho} \Big[\alpha^{1-\rho} + \delta (1-\alpha)^{1-\rho}\, \mathsf{E}\Big[(\pi' \tilde{R})^{1-\rho} f(X') \,|\, x \Big] \Big]. \end{aligned}$$

The $w^{1-\rho}/(1-\rho)$ terms cancel, confirming the guess with

$$f(x) = \alpha(x)^{1-\rho} + \delta(1-\alpha(x))^{1-\rho} \max_{\pi} \mathsf{E}\Big[(\pi' \tilde{R}(x))^{1-\rho} f(X') \,|\, X = x \Big],$$

where $\alpha(x)$ is determined by the FOC for consumption.

31. Same as Question 1.

    (a) Three equally likely states, two assets with prices $p_1 = p_2 = 1$, payoffs $(1,2,1)$ and $(0,1,3)$. State prices satisfy:

    $$q_1 + 2q_2 + q_3 = 1\,, \qquad q_2 + 3q_3 = 1\,.$$

    The one-dimensional family is $q = (-1 + 5q_3,\ 1 - 3q_3,\ q_3)$ for $q_3 \in (1/5,\ 1/3)$.

    (b) The SDF spanned by the assets is the same as computed in Question 1(b): $\widetilde{m} = (15/35,\ 33/35,\ 24/35)$.

32. **Risk premia and SDF covariances.**

Statement: If $\widetilde{m}$ is an SDF and there is a risk-free asset with return R_f, then for any return $\widetilde{R}$:

$$\mathsf{E}[\widetilde{R}] - R_f = -R_f\ \mathrm{cov}(\widetilde{m}, \widetilde{R})\,.$$

Definitions: An SDF is a random variable $\widetilde{m}$ such that $\mathsf{E}[\widetilde{m}\widetilde{R}] = 1$ for all returns $\widetilde{R}$. The risk premium of $\widetilde{R}$ is $\mathsf{E}[\widetilde{R}] - R_f$.

Proof: Since $\mathsf{E}[\widetilde{m}] = 1/R_f$ and $\mathsf{E}[\widetilde{m}\widetilde{R}] = 1$:

$$\mathrm{cov}(\widetilde{m}, \widetilde{R}) = \mathsf{E}[\widetilde{m}\widetilde{R}] - \mathsf{E}[\widetilde{m}]\mathsf{E}[\widetilde{R}] = 1 - \frac{\mathsf{E}[\widetilde{R}]}{R_f}\,.$$

Rearranging:

$$\mathsf{E}[\widetilde{R}] - R_f = -R_f\ \mathrm{cov}(\widetilde{m}, \widetilde{R})\,.$$

Thus, the risk premium depends on the covariance with the SDF. Assets that covary negatively with $\widetilde{m}$ (i.e., pay well in good states when $\widetilde{m}$ is low) have positive risk premia.

33. **MV frontier return as pricing factor.**

    Assume there are n risky assets with return vector $\tilde{\mathbf{R}}$, mean μ, covariance Σ, and a risk-free asset with return R_f. Let $\widetilde{R}_p$ be a mean-variance frontier return with $\widetilde{R}_p \neq R_f$.

    Definition: A return $\widetilde{R}_p$ is a pricing factor if $\mathsf{E}[\widetilde{R}_i] - R_f = \lambda \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p)$ for all returns $\widetilde{R}_i$ and some constant λ.

    A frontier return satisfies $\pi = \delta\Sigma^{-1}(\mu - R_f\iota)$ for some $\delta \neq 0$. For any asset i:

    $$\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p) = e_i'\Sigma\pi = \delta(\mu_i - R_f)\,,$$

    so $\mu_i - R_f = (1/\delta)\operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p)$. Setting $\lambda = 1/\delta = (\mathsf{E}[\widetilde{R}_p] - R_f)/\operatorname{var}(\widetilde{R}_p)$:

    $$\mathsf{E}[\widetilde{R}_i] - R_f = \frac{\mathsf{E}[\widetilde{R}_p] - R_f}{\operatorname{var}(\widetilde{R}_p)} \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_p)\,.$$

    Hence any frontier return (other than R_f) is a pricing factor.

34. Same as Questions 3 and 7.

    With CRRA ρ and $\log(\tilde{c}_1/c_0) \sim N(\mu, \sigma^2)$:

    $$\log R_f = -\log\delta + \rho\mu - \frac{\rho^2\sigma^2}{2}\,.$$

    See the derivation in Question 3.

35. Same as Question 14.

    With log utility ($u_h(w) = \log w$), Pareto optimal sharing rules are $\tilde{w}_h = b_h\tilde{w}_m$ where $b_h = \lambda_h/\sum_j \lambda_j > 0$.

    Proof: The FOC for the social planner's problem is $\lambda_h u_h'(\tilde{w}_h) = \tilde{\eta}$ for all h, i.e., $\lambda_h/\tilde{w}_h = \tilde{\eta}$. Thus $\tilde{w}_h = \lambda_h/\tilde{\eta}$. Summing: $\tilde{w}_m = (\sum_h \lambda_h)/\tilde{\eta}$, so $\tilde{\eta} = (\sum_h \lambda_h)/\tilde{w}_m$. Therefore

    $$\tilde{w}_h = \frac{\lambda_h}{\sum_j \lambda_j}\,\tilde{w}_m\,.$$

    Each investor's wealth is proportional to market wealth—the sharing rules are affine (in fact, linear).