Sample Final Exam Solutions

BUSI 521 / ECON 505 — Asset Pricing

Rice University

Question 1

Setup. An investor with infinite horizon, discount rate δ, and no labor income. Investment opportunities depend on a univariate Markov process X satisfying

$$\mathrm{d}X = \phi(X)\,\mathrm{d}t + \nu(X)'\,\mathrm{d}B.$$

There are n risky assets with expected returns μ, volatility matrix σ, and covariance matrix $\Sigma = \sigma\sigma'$, and a risk-free rate r. All may depend on X.

(a) HJB equation for the stationary value function $J(w, x)$.
The investor's value function is

$$J(w, x) = \max \mathsf{E}\left[\int_0^\infty \mathrm{e}^{-\delta t}\, u(C_t)\,\mathrm{d}t\right],$$

where the maximum is over consumption and portfolio processes. The HJB equation for the stationary value function $J(w, x)$ is:

$$0 = \max_{c,\pi}\left\{u(c) - \delta J + [rw + \pi'(\mu - r\iota)w - c]\,J_w + \frac{1}{2}\pi'\Sigma\pi\,w^2 J_{ww} + \phi\,J_x + \frac{1}{2}\nu'\nu\,J_{xx} + \pi'\sigma\nu\,w\,J_{wx}\right\}.$$

(b) Envelope condition.
The first-order condition with respect to c is

$$u'(c) = J_w.$$

This is the envelope condition: the marginal utility of consumption equals the marginal value of wealth.

(c) Optimal portfolio formula.
The first-order condition with respect to π is

$$(\mu - r\iota)\,w\,J_w + \Sigma\pi\,w^2 J_{ww} + \sigma\nu\,w\,J_{wx} = 0.$$

Solving for π:

$$\pi = -\frac{J_w}{wJ_{ww}}\Sigma^{-1}(\mu - r\iota) - \frac{J_{wx}}{wJ_{ww}}\Sigma^{-1}\sigma\nu.$$

(d) Log-optimal portfolio and hedging demands.
The first term,

$$\pi^{\text{log}} = -\frac{J_w}{wJ_{ww}}\Sigma^{-1}(\mu - r\iota),$$

is the *log-optimal portfolio* (also called the myopic or speculative demand). For a log-utility investor, $-J_w/(wJ_{ww}) = 1$, so this reduces to $\Sigma^{-1}(\mu - r\iota)$.
The second term,

$$\pi^{\text{hedge}} = -\frac{J_{wx}}{wJ_{ww}}\Sigma^{-1}\sigma\nu,$$

is the *hedging demand*. It hedges against changes in the investment opportunity set as captured by changes in X.

Question 2

Setup. Same as Question 1, but with log utility: $u(c) = \log c$. Conjecture $J(w, x) = f(x) + \frac{\log w}{\delta}$.

(a) Optimal consumption rate.
From the envelope condition, $u'(C) = J_w$, so

$$\frac{1}{C} = \frac{1}{\delta W}.$$

Hence $C = \delta W$.

(b) ODE for f.
With $J_w = 1/(\delta w)$, $J_{ww} = -1/(\delta w^2)$, and $J_{wx} = 0$, the optimal portfolio from the first-order condition becomes

$$\pi = \Sigma^{-1}(\mu - r\iota),$$

and the optimal consumption is $c = \delta w$. Substituting into the HJB equation:

$$0 = \log(\delta w) - \delta f - \log w + \frac{1}{\delta}\big[r + (\mu - r\iota)'\Sigma^{-1}(\mu - r\iota) - \delta\big] - \frac{1}{2\delta}(\mu - r\iota)'\Sigma^{-1}(\mu - r\iota)$$
$$+ \phi f_x + \frac{1}{2}\nu'\nu\, f_{xx}.$$

Using $\kappa^2 = (\mu - r\iota)'\Sigma^{-1}(\mu - r\iota)$ for the squared maximum Sharpe ratio and simplifying:

$$\phi(x)\, f'(x) + \frac{1}{2}\nu(x)'\nu(x)\, f''(x) = \delta f(x) - \frac{\log \delta}{\delta} - \frac{r - \delta + \kappa^2/2}{\delta}.$$

This is the ODE that f must satisfy (with r and κ depending on x).

(c) Derivation of the CAPM.
If the investor is a representative investor, then the market portfolio equals the investor's portfolio:

$$\pi_m = \Sigma^{-1}(\mu - r\iota).$$

Hence $\mu - r\iota = \Sigma\pi_m$. The return on the market portfolio W_m satisfies

$$\frac{\mathrm{d}W_m}{W_m} = [r + \pi_m'(\mu - r\iota)]\,\mathrm{d}t + \pi_m'\sigma\,\mathrm{d}B,$$

so the instantaneous covariance of any asset i's return with the market return is

$$\left(\frac{\mathrm{d}S_i}{S_i}\right)\left(\frac{\mathrm{d}W_m}{W_m}\right) = e_i'\Sigma\pi_m\,\mathrm{d}t.$$

From $\mu - r\iota = \Sigma\pi_m$, the risk premium of asset i is

$$\mu_i - r = e_i'\Sigma\pi_m = \frac{\mathrm{cov}(\mathrm{d}S_i/S_i,\ \mathrm{d}W_m/W_m)}{\mathrm{var}(\mathrm{d}W_m/W_m)}\big(\mu_m - r\big),$$

which is the continuous-time CAPM: expected excess returns are proportional to market beta.

Question 3

Setup. Aggregate consumption follows a geometric Brownian motion:

$$\frac{\mathrm{d}C}{C} = \alpha\,\mathrm{d}t + \theta\,\mathrm{d}B.$$

There is a representative investor with CRRA ρ and discount rate δ.

(a) SDF process and its dynamics.
The SDF process is the marginal rate of substitution:

$$M_t = \mathrm{e}^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-\rho}.$$

Using Itô's formula (specifically Exercise 12.2(c)):

$$\begin{aligned}\frac{\mathrm{d}M}{M} &= -\delta\,\mathrm{d}t - \rho\,\frac{\mathrm{d}C}{C} + \frac{\rho(1+\rho)}{2}\left(\frac{\mathrm{d}C}{C}\right)^2 \\ &= -\left(\delta + \rho\alpha - \frac{\rho(1+\rho)}{2}\theta^2\right)\mathrm{d}t - \rho\theta\,\mathrm{d}B.\end{aligned}$$

(b) Equilibrium risk-free rate.
The instantaneous risk-free rate equals minus the drift of $\mathrm{d}M/M$:

$$r = \delta + \rho\alpha - \frac{\rho(1+\rho)}{2}\theta^2.$$

(c) Market price-to-consumption ratio.
The market portfolio is a claim to the aggregate consumption stream. Its price is

$$P_t = \mathsf{E}_t\left[\int_t^\infty \frac{M_u}{M_t}\,C_u\,\mathrm{d}u\right].$$

Because C is a GBM and M depends only on C, we have $P_t = kC_t$ for a constant k. Computing:

$$\frac{M_u C_u}{M_t C_t} = \mathrm{e}^{-\delta(u-t)}\left(\frac{C_u}{C_t}\right)^{1-\rho}.$$

Since C_u/C_t is lognormal with $\log(C_u/C_t) \sim \mathcal{N}\big((\alpha - \theta^2/2)(u-t),\ \theta^2(u-t)\big)$,

$$\mathsf{E}_t\left[\frac{M_u C_u}{M_t C_t}\right] = \exp\left[\left(-\delta + (1-\rho)\alpha + \frac{(1-\rho)(-\rho)}{2}\theta^2\right)(u-t)\right].$$

This simplifies to $\mathrm{e}^{-\left(\delta-(1-\rho)\alpha+\rho(1-\rho)\theta^2/2\right)(u-t)}$. Define

$$\eta = \delta - (1-\rho)\alpha + \frac{\rho(\rho-1)}{2}\theta^2 = r + \rho\theta^2 - \alpha.$$

For $\eta > 0$ (needed for $k > 0$), the integral gives

$$k = \frac{1}{\eta} = \frac{1}{r + \rho\theta^2 - \alpha}.$$

(d) Risk premium of any asset.
For any asset with price S, the risk premium satisfies

$$(\mu_S - r)\,\mathrm{d}t = -\left(\frac{\mathrm{d}S}{S}\right)\left(\frac{\mathrm{d}M}{M}\right) = \rho\left(\frac{\mathrm{d}S}{S}\right)\left(\frac{\mathrm{d}C}{C}\right).$$

This is because $\mathrm{d}M/M$ has stochastic part $-\rho\theta\,\mathrm{d}B$, and the instantaneous covariance of any asset's return with consumption growth determines the risk premium.

(e) Name of the model.
This is the **Consumption CAPM (CCAPM)**, also known as the Breeden CCAPM.

Question 4

Setup. One risky asset, one Brownian motion B, constant risk-free rate r. The Sharpe ratio X_t (usually called λ) satisfies

$$dX = \kappa(\theta - X)\,dt + \nu\,dB.$$

Investor has CRRA ρ, discount rate δ. The market is complete, so M is unique.

(a) Optimal consumption from the Euler equation.
In a complete market, the first-order condition gives

$$e^{-\delta t} C_t^{-\rho} = \gamma M_t,$$

so

$$C_t = \left(\gamma e^{\delta t} M_t\right)^{-1/\rho},$$

where γ is the Lagrange multiplier on the static budget constraint.

(b) W_t/C_t as a function of X_t.
Optimal wealth is

$$W_t = \mathsf{E}_t\left[\int_t^\infty \frac{M_u}{M_t} C_u\,du\right].$$

Substituting $C_u = (\gamma e^{\delta u} M_u)^{-1/\rho}$:

$$\frac{W_t}{C_t} = \mathsf{E}_t\left[\int_t^\infty \frac{M_u}{M_t} \cdot \frac{C_u}{C_t}\,du\right] = \mathsf{E}_t\left[\int_t^\infty e^{-(\delta/\rho)(u-t)} \left(\frac{M_u}{M_t}\right)^{1-1/\rho} du\right].$$

The SDF process satisfies $dM/M = -r\,dt - X\,dB$, so M_u/M_t depends only on the path of X and B from t onward. Since X is Markov, the conditional expectation depends only on X_t. Hence $W_t/C_t = f(X_t)$ for some function f, and $W_t = C_t f(X_t)$.

(c) Optimal portfolio.
We have $W = C\,f(X)$, so

$$\frac{dW}{W} = \frac{dC}{C} + \frac{df}{f} + \left(\frac{dC}{C}\right)\left(\frac{df}{f}\right).$$

The stochastic part of dC/C comes from

$$\frac{dC}{C} = -\frac{1}{\rho}\frac{dM}{M} + \cdots = \frac{1}{\rho} X\,dB + \cdots$$

The stochastic part of df/f is $(f'/f)\nu\,dB$. Therefore, the stochastic part of dW/W is

$$\left(\frac{X}{\rho} + \frac{f'}{f}\nu + \frac{f'}{f} \cdot \frac{\nu X}{\rho}\right) dB.$$

However, since $(dC/C)(df/f)$ contributes $(\nu X/\rho)(f'/f)\,dt$ (a drift term), the stochastic part of dW/W is

$$\left(\frac{X}{\rho} + \frac{f'(X)}{f(X)}\nu\right) dB.$$

Setting $\pi\sigma\,dB$ equal to this:

$$\pi = \frac{1}{\sigma}\left(\frac{X}{\rho} + \frac{f'(X)}{f(X)}\nu\right).$$

The first term $X/(\rho\sigma) = \lambda/(\rho\sigma) = (\mu - r)/(\rho\sigma^2)$ is the **log-optimal (myopic) portfolio**. The second term $\nu f'(X)/(\sigma f(X))$ is the **hedging demand**—it hedges against changes in investment opportunities driven by changes in X.

(d) ODE for f.
From part (b), $W_t = C_t f(X_t)$ and

$$\int_0^t M_u C_u \, \mathrm{d}u + M_t C_t f(X_t)$$

is a martingale. Its differential is

$$M_t C_t \, \mathrm{d}t + \mathrm{d}\big[M_t C_t f(X_t)\big] = 0 \quad \text{(in the d}t\text{ terms)}.$$

We can write $M_t C_t = M_t^{1-1/\rho} \cdot (\mathrm{const} \cdot \mathrm{e}^{-\delta t/\rho})$. Setting the drift of the martingale to zero and using the dynamics of M and X yields:

$$\frac{1}{2}\nu^2 f''(x) + \left[\kappa(\theta - x) + \frac{(\rho-1)}{\rho}\nu x\right] f'(x) + \left[\frac{(\rho-1)r}{\rho} + \frac{(\rho-1)x^2}{2\rho^2} - \frac{\delta}{\rho}\right] f(x) + 1 = 0.$$

This is the ODE that f must satisfy.

Question 5

Setup. Log-utility investor in a continuous-time model with a single state variable X. Value function $J(w,x) = a\log w + f(x)$.

Optimal portfolio.
With $J_w = a/w$, $J_{ww} = -a/w^2$, and $J_{wx} = 0$, the first-order condition for π gives

$$\pi = \Sigma^{-1}(\mu - r\iota).$$

This is the myopic (log-optimal) portfolio. There is no hedging demand because $J_{wx} = 0$.

Optimal consumption.
From $u'(c) = J_w$: $1/c = a/w$, so $c = w/a$. Substituting into the HJB equation with δ as the discount rate:

$$0 = \log(w/a) - \delta a \log w - \delta f + a\left[r + \kappa^2 - \frac{1}{a}\right] - \frac{a}{2}\kappa^2 + a\left[r + \kappa^2 - \frac{1}{a}\right]$$

After careful substitution, the HJB equation becomes:

$$0 = \log w - \log a - \delta a \log w - \delta f + a\left(r - \frac{1}{a}\right) + \frac{a}{2}\kappa^2 + \phi f' + \frac{1}{2}\nu'\nu\, f''.$$

For the $\log w$ terms to cancel, we need $1 = \delta a$, i.e., $a = 1/\delta$.

ODE for f.
Substituting $a = 1/\delta$ and collecting terms, the ODE for f is:

$$\phi(x)\, f'(x) + \frac{1}{2}\nu(x)'\nu(x)\, f''(x) = \delta f(x) - \frac{\log\delta}{\delta} - \frac{r(x) - \delta + \kappa(x)^2/2}{\delta},$$

where $\kappa^2 = (\mu - r\iota)'\Sigma^{-1}(\mu - r\iota)$ is the squared maximum Sharpe ratio (which may depend on x). This is the same ODE as in Question 2(b).

Question 6

Setup. The price S of a non-dividend-paying asset is a geometric Brownian motion:

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B,$$

with a constant risk-free rate r. A security pays at date T:

$$g(S_T) = \begin{cases} K_1 & \text{if } S_T < K_1, \\ S_T & \text{if } K_1 \le S_T \le K_2, \\ K_2 & \text{if } S_T > K_2. \end{cases}$$

PDE for the value.
Let $V(t,S)$ denote the value of the security at date t when the asset price is S. Under the risk-neutral probability, S satisfies

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*,$$

where B^* is a Brownian motion under the risk-neutral measure. The fundamental PDE is obtained by writing the rate of return $(\mathrm{d}V)/V$ and equating its expected value under the risk-neutral probability to r:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV,$$

with the terminal condition $V(T,S) = g(S_T)$.

Pricing using an SDF process.
An SDF process M satisfies

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B,$$

where $\lambda = (\mu - r)/\sigma$ is the market price of risk. By Girsanov's theorem, $\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t$ defines a Brownian motion under the risk-neutral probability.
The value of the security at date t is

$$V_t = \mathsf{E}_t\left[\frac{M_T}{M_t}\, g(S_T)\right].$$

Equivalently, using the risk-neutral probability:

$$V_t = \mathrm{e}^{-r(T-t)}\, \mathsf{E}_t^*\big[g(S_T)\big],$$

where E^* denotes expectation under the risk-neutral measure. Under this measure, $\log S_T$ is normally distributed:

$$\log S_T = \log S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(B_T^* - B_t^*).$$

The payoff $g(S_T) = K_1 \cdot \mathbf{1}_{S_T < K_1} + S_T \cdot \mathbf{1}_{K_1 \le S_T \le K_2} + K_2 \cdot \mathbf{1}_{S_T > K_2}$ can be decomposed as:

$$g(S_T) = K_1 + \max(S_T - K_1, 0) - \max(S_T - K_2, 0).$$

This is a portfolio of K_1 in cash, a long call with strike K_1, and a short call with strike K_2. Each call can be priced using the Black–Scholes formula, giving

$$V_t = \mathrm{e}^{-r(T-t)} K_1 + \mathrm{BS}_{\text{call}}(S_t, K_1, T-t) - \mathrm{BS}_{\text{call}}(S_t, K_2, T-t).$$

Question 7

Setup. A single risky asset with $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$, no dividends prior to T, constant r. A security pays S_T^2 at date T.

(a) Price using the SDF process.
The SDF process satisfies $\mathrm{d}M/M = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B$ with $\lambda = (\mu - r)/\sigma$. The price at date t is

$$V_t = \mathsf{E}_t\left[\frac{M_T}{M_t}\,S_T^2\right] = \mathrm{e}^{-r(T-t)}\,\mathsf{E}_t^*\left[S_T^2\right].$$

Under the risk-neutral probability, S satisfies $\mathrm{d}S/S = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$, so

$$S_T = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma(B_T^* - B_t^*)\right].$$

Therefore

$$S_T^2 = S_t^2 \exp\left[\left(2r - \sigma^2\right)(T-t) + 2\sigma(B_T^* - B_t^*)\right].$$

Taking the risk-neutral expectation (using $\mathsf{E}^*[\mathrm{e}^{aZ}] = \mathrm{e}^{a^2/2}$ for $Z \sim \mathcal{N}(0,1)$):

$$\mathsf{E}_t^*[S_T^2] = S_t^2 \exp\left[(2r - \sigma^2)(T-t) + 2\sigma^2(T-t)\right] = S_t^2\,\mathrm{e}^{(2r+\sigma^2)(T-t)}.$$

Hence

$$\boxed{V_t = S_t^2\,\mathrm{e}^{(r+\sigma^2)(T-t)}.}$$

(b) Fundamental PDE.
Let $V = f(t, S)$ denote the price. The fundamental PDE (derived by equating the expected return under the risk-neutral probability to the risk-free rate) is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf,$$

with boundary condition $f(T, S) = S^2$.
We can verify: $f(t, S) = S^2\,\mathrm{e}^{(r+\sigma^2)(T-t)}$ gives

$$\begin{aligned} f_t &= -(r+\sigma^2)f, \\ f_S &= 2S\,\mathrm{e}^{(r+\sigma^2)(T-t)}, \\ f_{SS} &= 2\,\mathrm{e}^{(r+\sigma^2)(T-t)}. \end{aligned}$$

Substituting:

$$-(r+\sigma^2)f + 2rS^2\mathrm{e}^{(r+\sigma^2)(T-t)} + \sigma^2 S^2 \mathrm{e}^{(r+\sigma^2)(T-t)} = -(r+\sigma^2)f + (2r+\sigma^2)f = rf. \checkmark$$

Question 8

Setup. A perpetual call option on an asset with constant dividend yield $q > 0$, constant volatility σ, and constant risk-free rate r. Under the risk-neutral probability,

$$\frac{\mathrm{d}S}{S} = (r-q)\,\mathrm{d}t + \sigma\,\mathrm{d}B^*.$$

Solution.
Consider exercising the call when S first rises to a level $s^* > K$. For $S < s^*$, the value of the option is $f(S)$, which satisfies the ODE (obtained by equating the expected return under the risk-neutral measure to r):

$$\frac{1}{2}\sigma^2 S^2 f''(S) + (r-q)Sf'(S) = rf(S).$$

The general solution is $f(S) = aS^\beta + bS^{-\gamma}$, where $\beta > 1$ and $-\gamma < 0$ are the two roots of the characteristic equation

$$\frac{1}{2}\sigma^2 z(z-1) + (r-q)z - r = 0.$$

Equivalently, $\frac{1}{2}\sigma^2 z^2 + (r - q - \frac{1}{2}\sigma^2)z - r = 0$.
Boundary conditions:

- As $S \to 0$, $f(S) \to 0$, which requires $b = 0$. So $f(S) = aS^\beta$.
- **Value matching:** $f(s^*) = s^* - K$, giving $a(s^*)^\beta = s^* - K$, i.e., $a = (s^* - K)/(s^*)^\beta$.

Substituting, the option value for any $S \le s^*$ is

$$f(S) = (s^* - K)\left(\frac{S}{s^*}\right)^\beta.$$

Optimal exercise boundary by direct optimization. Maximize $f(S)$ over s^* (for fixed S). Taking the derivative:

$$\frac{\partial f}{\partial s^*} = \left(\frac{S}{s^*}\right)^\beta \left[1 - \frac{\beta(s^* - K)}{s^*}\right] = 0.$$

Since $(S/s^*)^\beta > 0$, we need $s^* - \beta(s^* - K) = 0$, which gives

$$s^* = \frac{\beta K}{\beta - 1}.$$

The value of the perpetual call for $S \le s^*$ is

$$f(S) = (s^* - K)\left(\frac{S}{s^*}\right)^\beta,$$

where $s^* = \beta K/(\beta - 1)$ and $\beta > 1$ is the positive root of the characteristic equation.
The optimal exercise policy is: exercise the first time S reaches $s^* = \beta K/(\beta - 1)$.

Question 9

Setup. An asset with price S, constant volatility σ, and constant dividend yield δ. You have the option to sell at price K at any time τ. The payoff is $h(S_\tau) = K - S_\tau$. Consider exercising when S first falls to $s^* \le S_0$. Let $f(s)$ be the option value for $s^* \le s \le S_0$.

(a) ODE for f.
Under the risk-neutral probability, $\mathrm{d}S/S = (r - \delta)\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$. Since the option has no cash flow before exercise, its expected return under the risk-neutral probability must equal r. This gives the ODE:

$$\frac{1}{2}\sigma^2 s^2 f''(s) + (r - \delta)\, s\, f'(s) - r\, f(s) = 0.$$

(b) Power-function solution.
Try $f(s) = as^\beta$. Then $f'(s) = a\beta s^{\beta-1}$ and $f''(s) = a\beta(\beta - 1)s^{\beta-2}$. Substituting:

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r = 0.$$

This is the characteristic (quadratic) equation for β. It has two roots: one positive $\beta > 0$ and one negative $-\gamma < 0$.

(c) Boundary condition and value.
As $s \to \infty$, the put option value should vanish: $f(s) \to 0$. The positive root β would give $f(s) \to \infty$ as $s \to \infty$, so we set its coefficient to zero. Hence

$$f(s) = a\, s^{-\gamma},$$

where $-\gamma$ is the negative root.
Value matching at $s = s^*$: $f(s^*) = K - s^*$, so

$$a(s^*)^{-\gamma} = K - s^* \quad \Longrightarrow \quad a = (K - s^*)(s^*)^\gamma.$$

Thus

$$f(s) = (K - s^*)\left(\frac{s^*}{s}\right)^\gamma.$$

(d) Optimal exercise boundary by direct optimization.
Maximize $f(s)$ over s^* (for fixed $s > s^*$). Writing $f = (K - s^*)(s^*)^\gamma s^{-\gamma}$ and differentiating with respect to s^*:

$$\frac{\partial f}{\partial s^*} = s^{-\gamma}(s^*)^{\gamma-1}[\gamma(K - s^*) - s^*] = 0.$$

Since $s^{-\gamma}(s^*)^{\gamma-1} > 0$, we need $\gamma(K - s^*) = s^*$, which gives

$$s^* = \frac{\gamma K}{1 + \gamma}.$$

Since $\gamma > 0$, we have $s^* < K$ as required.
The optimal exercise policy is: exercise the first time S falls to

$$s^* = \frac{\gamma K}{1 + \gamma},$$

and the value of the perpetual put for $s \ge s^*$ is

$$f(s) = (K - s^*)\left(\frac{s^*}{s}\right)^\gamma = \frac{K}{1 + \gamma}\left(\frac{\gamma K}{(1 + \gamma)s}\right)^\gamma.$$