Chapter 19: Perpetual Options

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Set-up

- Single risky asset with price S and constant volatility σ, single Brownian motion, constant risk-free rate
- Dividend paid by risky asset in time period $\mathrm{d}t$ is $qS_t\,\mathrm{d}t$ for constant q ("dividend yield")
- Total return is
$$\frac{\mathrm{d}S + qS\,\mathrm{d}t}{S} = \frac{\mathrm{d}S}{S} + q\,\mathrm{d}t$$
- Total expected return under RNP is risk-free rate, so
$$\frac{\mathrm{d}S}{S} = (r - q)\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$$
for a risk-neutral Brownian motion B^*.

Perpetual Call

- Perpetual call option with strike K
- Why exercise? To capture the dividend. But the asset price and dividend must be high enough before it is optimal to do so.
- An example of a strategy is to pick a number x and exercise the first time S_t gets up to x. The optimal strategy will be of this type.
- The problem of finding the optimal exercise time is in the class of problems often called optimal stopping.

Exercise Boundary

- We will first calculate the value if we exercise the first time S_t gets up to x for an arbitrary $x > S_0$.
- Let $\tau = \inf\{t \mid S_t \geq x\}$. This is called the hitting time of x.
- By the time-homogeneity of S, the value at any $t < \tau$ depends only on S_t. Call it $f(S_t)$.
- More formally,

$$f(s) = \mathsf{E}^*[\mathrm{e}^{-r\tau}((x - K) \mid S_0 = s] = \mathsf{E}^*[\mathrm{e}^{-r(\tau-t)}(x - K) \mid S_t = s]$$

Fundamental ODE

- The fundamental ODE is

$$\frac{\text{drift}^* \text{ of } f}{f} = r$$

which is

$$(r-q)Sf' + \frac{1}{2}\sigma^2 S^2 f'' = rf$$

- Trying a power solution $f(S) = S^\gamma$, we see that f satisfies the ODE if and only if

$$(r-q)\gamma + \frac{1}{2}\sigma^2\gamma(\gamma-1) = r$$

- The quadratic formula shows that there are two real roots of this equation. One is negative and the other is greater than 1.

General Solution and Boundary Conditions

- Let $\gamma =$ absolute value of negative root, and $\beta =$ positive root.
- The general solution of the ODE is

$$aS^{-\gamma} + bS^{\beta}$$

  for constants a and b that must be determined by boundary conditions.
- The value f of the call exercised at the hitting time of x satisfies $f(0) = 0$ and $f(x) = x - K$. The condition $f(0) = 0$ implies $a = 0$, and the condition $f(x) = x - K$ implies $b = (x - K)x^{-\beta}$.
- The value of the call is

$$f(S_t) = (x - K)\left(\frac{S_t}{x}\right)^{\beta}$$

Optimal Stopping

- To optimize, maximize $(x - K)\left(\frac{S_t}{x}\right)^\beta$ over x.
- The factor S_t^β is a positive constant and is irrelevant for determining the optimum, so we can maximize

$$(x - K)x^{-\beta} = x^{1-\beta} - Kx^{-\beta}$$

- The FOC is

$$(1-\beta)x^{-\beta} + \beta K x^{-\beta-1} = 0$$

- Equivalently,

$$(1-\beta)x + \beta K = 0$$

So,

$$x^* = \frac{\beta}{\beta - 1}K$$

Perpetual Put

- Recall that the general solution of the ODe is $f(s) = as^{-\gamma} + bs^{\beta}$.
- For a put, we exercise the first time S_t drops to a boundary x.
- The boundary conditions for a put are $f(\infty) = 0$, and $f(x) = K - x$. The condition $f(\infty) = 0$ implies $b = 0$. The condition $f(x) = K - x$ implies $a = (K - x)x^{\gamma}$.
- So, the put value is

$$f(S_t) = (K - x)\left(\frac{x}{S_t}\right)^{\gamma}$$

- The FOC for maximizing over x is

$$\gamma K x^{\gamma - 1} - (1 + \gamma)x^{\gamma} = 0$$

Maximizing over x yields $x^* = \gamma K/(1 + \gamma)$.