

Chapter 15: Continuous-Time Topics

Kerry Back
BUSI 521/ECON 505
Spring 2024
Rice University

Risk Neutral Probabilities

Risk-Neutral Probabilities in Continuous Time

- Consider $T < \infty$.
- Let R denote the money market account price with $R_0 = 1$.
- Let M be an SDF process. Assume MR is a martingale so

$$E[M_T R_T] = R_0 = 1$$

- Define

$$\mathbb{Q}(A) = E[M_T R_T 1_A]$$

for each event A that is distinguishable at date T , where $1_A = 1$ when the state of the world is in A and 0 otherwise.

- It follows that \mathbb{Q} is a probability (measure) and

$$E^*[X_T] = E[M_T R_T X_T]$$

for any random variable X_T depending on date- T information, where E^* denotes expectation with respect to \mathbb{Q} .

Risk-Neutral Valuation

- Let W be such that MW is a martingale under the physical probability. Because we changed the probability using MR , a theorem in probability theory tells us that

$$\frac{MW}{MR}$$

is a \mathbb{Q} -martingale.

- So, W/R is a \mathbb{Q} -martingale. Thus,

$$W_t = R_t E_t^* \left[\frac{W_T}{R_T} \right] = E_t^* \left[\exp \left(- \int_t^T r_u du \right) W_T \right].$$

- In other words, asset values are expected discounted values, taking expectations with respect to the risk neutral probability and discounting at the instantaneous risk-free rate.
- It follows that expected returns under the RNP equal the risk-free rate.

Girsanov's Theorem

- Let M be an SDF process with

$$\frac{dM}{M} = -r dt - \lambda' dB$$

Here, r and λ can be stochastic processes.

- Define the risk-neutral probability \mathbb{Q} using the martingale MR .
- The vector B is not a vector of Brownian motions under \mathbb{Q}
 - Its drift is nonzero.
 - But, we still have quadratic variation $(dB)(dB)' = I dt$, so it is “close” to being a vector of Brownian motions.
- Girsanov's theorem states that B^* defined by $B_0^* = 0$ and

$$dB^* = dB + \lambda dt$$

is a vector of independent Brownian motions under the risk-neutral probability \mathbb{Q} .

Asset Returns under a Risk-Neutral Probability

- Recall that the vector of asset returns is

$$\frac{dS}{S} = \mu dt + \sigma dB$$

- Define $dB^* = dB + \lambda dt$. Substitute to obtain

$$\begin{aligned}\frac{dS}{S} &= \mu dt + \sigma (dB^* - \lambda dt) \\ &= (\mu - \sigma\lambda) dt + \sigma dB^* \\ &= r_f dt + \sigma dB^*\end{aligned}$$

Fundamental PDE

Fundamental PDE

- Suppose S is a univariate GBM that is a dividend-reinvested price:
 $dS/S = \mu dt + \sigma dB = r dt + \sigma dB^*$.
- Let $f(t, S_t)$ denote the value at t of some at $T > t$ that depends on S_T .
- The risk-neutral expected rate of return is the risk-free rate, so

$$\frac{\text{drift of } f \text{ under RNP}}{f(t, S_t)} = r$$

Fundamental PDE

- Suppose S is a univariate GBM that is a dividend-reinvested price:
 $dS/S = \mu dt + \sigma dB = r dt + \sigma dB^*$.
- Let $f(t, S_t)$ denote the value at t of some at $T > t$ that depends on S_T .
- The risk-neutral expected rate of return is the risk-free rate, so

$$\frac{\text{drift of } f \text{ under RNP}}{f(t, S_t)} = r$$

$$\begin{aligned} df &= f_t dt + f_S dS + \frac{1}{2} f_{SS} (dS)^2 \\ &= f_t dt + f_S (rS dt + \sigma S dB^*) + \frac{1}{2} f_{SS} (S^2 \sigma^2 dt) \end{aligned}$$

Fundamental PDE

- Suppose S is a univariate GBM that is a dividend-reinvested price:
 $dS/S = \mu dt + \sigma dB = r dt + \sigma dB^*$.
- Let $f(t, S_t)$ denote the value at t of some at $T > t$ that depends on S_T .
- The risk-neutral expected rate of return is the risk-free rate, so

$$\frac{\text{drift of } f \text{ under RNP}}{f(t, S_t)} = r$$

$$\begin{aligned} df &= f_t dt + f_S dS + \frac{1}{2} f_{SS} (dS)^2 \\ &= f_t dt + f_S (rS dt + \sigma S dB^*) + \frac{1}{2} f_{SS} (S^2 \sigma^2 dt) \end{aligned}$$

- So

$$\text{drift of } f \text{ under RNP} = f_t + rSf_S + \frac{1}{2} \sigma^2 S^2 f_{SS}$$

The fundamental PDE is

$$f_t + rSf_S + \frac{1}{2}\sigma^2 S^2 f_{SS} = rf \quad (*)$$