Chapter 13: Continuous-Time Markets

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Securities Market Model

Notation

- Money market account has price R with $\mathrm{d}R/R = r\,\mathrm{d}t$.
- n locally risky assets with dividend-reinvested prices S_i.
- $\mu =$ vector of n stochastic processes μ_i
- $\sigma = n \times k$ matrix of stochastic processes
- $B =$ vector of k independent Brownian motions. $k \geq n$.
- Assume no redundant assets, meaning σ has rank n.

Asset Price Dynamics

- Assume, for each risky asset i,

$$\frac{\mathrm{d}S_{it}}{S_{it}} = \mu_{it}\,\mathrm{d}t + \sum_{j=1}^{k} \sigma_{ijt}\,\mathrm{d}B_{jt}$$

- Stacking the asset returns,

$$\mathrm{d}S/S \stackrel{\text{def}}{=} \begin{pmatrix} \mathrm{d}S_{1t}/S_{1t} \\ \vdots \\ \mathrm{d}S_{nt}/S_{nt} \end{pmatrix} = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$$

Covariance Matrix of Returns

- Drop the t subscript for simplicity. We have

$$\left(\frac{\mathrm{d}S_i}{S_i}\right)\left(\frac{\mathrm{d}S_\ell}{S_\ell}\right) = \left(\sum_{j=1}^{k} \sigma_{ij}\,\mathrm{d}B_j\right)\left(\sum_{j=1}^{k} \sigma_{\ell j}\,\mathrm{d}B_j\right)$$
$$= \sum_{j=1}^{k} \sigma_{ij}\sigma_{\ell j}\,\mathrm{d}t$$

- Stacking the returns:

$$(\mathrm{d}S/S)\left(\frac{\mathrm{d}S}{S}\right)' = (\sigma\,\mathrm{d}B)(\sigma\mathrm{d}B)'$$
$$= \sigma\,(\mathrm{d}B)(\mathrm{d}B)'\sigma' = \sigma\sigma'\,\mathrm{d}t = \Sigma\,\mathrm{d}t$$

for $\Sigma = \sigma\sigma'$.

Intertemporal Budget Constraint

- Let ϕ_i denote the amount of the consumption good invested in risky asset i.
- Let $W =$ wealth, $C =$ consumption, $Y =$ labor income.
- The intertemporal budget constraint is

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + \theta'\,\mathrm{d}S + (W - \theta' S) r\,\mathrm{d}t$$

  where $\theta = (\theta_1, \ldots, \theta_n)'$ denotes share holdings.
- Setting $\phi_i = \theta_i S_i \Rightarrow$

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + \phi'\,(\mathrm{d}S/S) + (W - \phi'\iota) r\,\mathrm{d}t$$

- Equivalently,

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + rW\,\mathrm{d}t + \phi'(\mathrm{d}S/S - r\iota)\,\mathrm{d}t$$

- Equivalently,

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + rW\,\mathrm{d}t + \phi'(\mu - r\iota)\,\mathrm{d}t + \phi'\sigma\,\mathrm{d}B$$

In Terms of Fractions of Wealth Invested

- Assuming $W > 0$, we can define $\pi = \phi/W$ and write the intertemporal budget constraint as

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + rW\,\mathrm{d}t + W\pi'(\mu - r\iota)\,\mathrm{d}t + W\pi'\sigma\,\mathrm{d}B$$

- Equivalently,

$$\frac{\mathrm{d}W}{W} = \frac{Y - C}{W}\,\mathrm{d}t + r\,\mathrm{d}t + \pi'(\mu - r\iota)\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}B$$

- If $Y = C$, the wealth process is said to be self financing.

First Optimization Problem

- Horizon T. No intermediate consumption ($C = 0$). No labor income ($Y = 0$). Log utility for terminal wealth. W_0 given.
- max $\mathsf{E}[\log(W_T)]$ over portfolio processes π subject to

$$\frac{\mathrm{d}W}{W} = r\,\mathrm{d}t + \pi'(\mu - r\iota)\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}B$$

- Solve the wealth equation like we solved for GBM before (take logs, integrate, then exponentiate). We get

$$W_T = W_0 \exp\left(\int_0^T \left(r_t + \pi_t'(\mu_t - r_t) - \frac{1}{2}\pi_t'\Sigma_t\pi_t\right)\,\mathrm{d}t + \int_0^T \pi_t'\sigma_t\,\mathrm{d}B_t\right)$$

- So, $\mathsf{E}[\log W_T]$ is

$$\log W_0 + \mathsf{E}\left[\int_0^T \left(r_t + \pi_t'(\mu_t - r_t) - \frac{1}{2}\pi_t'\Sigma_t\pi_t\right)\,\mathrm{d}t + \int_0^T \pi_t'\sigma_t\,\mathrm{d}B_t\right]$$

- Use iterated expectations to get

$$\log W_0 + \mathsf{E}_T\left[\int_0^T \mathsf{E}_t\left[r_t + \pi_t'(\mu_t - r_t) - \frac{1}{2}\pi_t'\Sigma_t\pi_t\right]\mathrm{d}t + \int_0^T \mathsf{E}_t[\pi_t'\sigma_t\,\mathrm{d}B_t]\right]$$

- Actually need a technical condition for this:

$$\mathsf{E}\int_0^T \pi_t'\Sigma_t\pi_t\,\mathrm{d}t < \infty$$

  which implies a local martingale is a martingale.

- Conclusion is: choose π_t to maximize

$$\pi_t'(\mu_t - r_t) - \frac{1}{2}\pi_t'\Sigma_t\pi_t$$

- Implies

$$\pi_t^* = \Sigma_t^{-1}(\mu_t - r_t)$$

SDF Processes

Definition of SDF Processes

- Define a stochastic process M to be an SDF process if
  - $M_0 = 1$
  - $M_t > 0$ for all t with probability 1
  - MR is a local martingale, where R denotes the price of the money market account,
  - MS_i is a local martingale, for $i = 1, \ldots, n$, where the S_i are the dividend-reinvested asset prices.
- 'Local martingale' means zero drift (no $\mathrm{d}t$ part).

Characerization of SDF Processes

- We can show: A stochastic process $M > 0$ with $M_0 = 1$ is an SDF process if and only if $\mathsf{E}[\mathrm{d}M/M] = -r\,\mathrm{d}t$ and

$$(\mu - r\iota)\,\mathrm{d}t \;=\; -(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right)$$

- Use $MR =$ local martingale to get $\mathsf{E}[\mathrm{d}M/M] = -r\,\mathrm{d}t$.
- Use $MS_i =$ local martingale for each i to get displayed equation.

No Uncertainty or Risk Neutrality

- SDF process is
$$M_t = \mathrm{e}^{-rt}$$
if r is constant or
$$M_t = \mathrm{e}^{-\int_0^t r_s \, \mathrm{d}s}$$
if r varies over time.
- So,
$$\frac{\mathrm{d}M}{M} = -r \, \mathrm{d}t$$
- With risk aversion, it is only true that the drift of $\mathrm{d}M/M$ is $-r$ which we express as $\mathsf{E}[\mathrm{d}M/M] = -r \, \mathrm{d}t$

Single Period Model

- The condition $E[\mathrm{d}M/M] = -r\,\mathrm{d}t$ parallels a single period model. Set $M_0 = 1$ and $M_1 = \tilde{m}$. Then,
  - $\Delta M/M_0 = (\tilde{m} - 1)/1$
  - $\mathsf{E}[\Delta M/M_0] = 1/R_f - 1 = (1 - R_f)/R_f = -r_f/R_f$
- The condition

$$(\mu - r\iota)\,\mathrm{d}t \;=\; -(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right)$$

parallels

$$(\forall\, i) \quad \mathsf{E}[\widetilde{R}_i] - R_f = -R_f\,\mathrm{cov}(\widetilde{R}_i, \tilde{m})$$

Prices of Risk

- Start with M being an Itô process with drift of $\mathrm{d}M/M$ being $-r$. This means
$$\frac{\mathrm{d}M_t}{M_t} = -r_t\,\mathrm{d}t - \lambda_t'\,\mathrm{d}B_t$$
for some λ process.
- The choice of $-\lambda$ instead of $+\lambda$ is arbitrary but convenient.
- Then,
$$(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right) = -\sigma(\mathrm{d}B)(\mathrm{d}B)'\lambda = \sigma\lambda\,\mathrm{d}t$$
- So,
$$(\mu - r\iota)\,\mathrm{d}t = -(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right) \quad \Rightarrow \quad \mu - r = \sigma\lambda$$
- λ called price of risk process.

Projections of SDF Processes

- One solution λ of the equation $\sigma\lambda = \mu - r\iota$ is

$$\lambda_p \stackrel{\text{def}}{=} \sigma'(\sigma\sigma')^{-1}(\mu - r\iota) = \sigma'\Sigma^{-1}(\mu - r\iota)$$

- For this solution,

$$\begin{aligned} \lambda_p' \, \mathrm{d}B &= (\mu - r\iota)'\Sigma^{-1}\sigma \, \mathrm{d}B \\ &= \pi'\sigma \, \mathrm{d}B \end{aligned}$$

  for $\pi = \Sigma^{-1}(\mu - r\iota)$ (the log-optimal portfolio). Thus, it is spanned by the assets.

- Every solution λ of the equation $\sigma\lambda = \mu - r\iota$ is of the form

$$\lambda = \lambda_p + \zeta$$

  where ζ is orthogonal to the assets in the sense that $\sigma\zeta = 0$.

Valuation

Valuation

- For an asset with price process P and dividend process D,

$$P_t = \mathsf{E}_t\left[\int_t^u \frac{M_\tau}{M_t} D_\tau \,\mathrm{d}\tau + \frac{M_u}{M_t} P_u\right]$$

  for any SDF process M (subject to a local martingale being a martingale).
- Ruling out bubbles, we can take u to infinity.
- Likewise, for any (W, C) satisfying the intertemporal budget constraint (assuming a local martingale is a martingale),

$$W_t = \mathsf{E}_t\left[\int_t^u \frac{M_\tau}{M_t} (C_\tau \ - Y_\tau), \mathrm{d}\tau + \frac{M_u}{M_t} W_u\right]$$

- Ruling out Ponzi schemes, we can take u to infinity.

Complete Markets

How Many Assets do we Need?

- Assume the Brownian motions are the only sources of uncertainty.
- Then the market is complete if the rank of σ is k (as many non-redundant assets as there are Brownian motions).
- We are assuming for simplicity that there are no redundant asets (rank σ is n), so completeness is equivalent to σ being square and nonsingular.

Why Completeness?

- Martingale representation theorem: with Brownian uncertainty, every martingale Y is spanned by the Brownian motions meaning $\mathrm{d}Y = \gamma' \, \mathrm{d}B$.
- When σ is square and nonsingular, we can set $\pi = \sigma^{-1}\gamma$ to get $\mathrm{d}Y = \pi' \sigma \, \mathrm{d}B$ w, which is the stochastic part of a portfolio return.

Uniqueness of the SDF Process

- When markets are complete, there is a unique solution of $\sigma\lambda = \mu - r\iota$ given by $\lambda = \sigma^{-1}(\mu - r)$.
- So, there is a unique SDF process

Second Optimization Problem

- Complete markets, finite horizon, continuous consumption, no labor income. Consumption process must satisfy

$$W_0 = \mathsf{E}\int_0^T M_t C_t \, \mathrm{d}t$$

- max

$$\mathsf{E}\int_0^T \mathrm{e}^{-\delta t} u(C_t) \, \mathrm{d}t$$

  subject to the above constraint.

- Lagrangean:

$$\mathsf{E}\int_0^T \left\{ \mathrm{e}^{-\delta t} u(C_t) - \gamma M_T C_t \right\} \, \mathrm{d}t$$

- Maximize pointwise. FOC is

$$u'(C_t) = \gamma M_t$$