Chapter 2: Portfolio Choice

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Simplest Problem

- Single risky asset, price p per share at date 0, price $\tilde{x}$ per share at date 1.
- Risk-free asset with interest rate r_f.
- Investor has w_0 to invest. Allocates between risk-free and risky assets.
- Let $\theta =$ number of shares of risky asset. Then $w_0 - p\theta$ is invested risk-free (this could be negative, meaning borrowing).
- θ is chosen to maximize

$$\mathsf{E}\big[u\big(\theta\tilde{x} + (w_0 - p\theta)(1 + r_f)\big)\big]$$

- FOC is

$$\mathsf{E}\big[u'\big(\theta^*\tilde{x} + (w_0 - p\theta^*)(1 + r_f)\big)\big\{\tilde{x} - p(1 + r_f)\big\}\big] = 0$$

More on the FOC

- Date-1 wealth is

$$\tilde{w}^* := \theta^* \tilde{x} + (w_0 - p\theta^*)(1 + r_f)$$

- Divide the FOC by p. Set $\widetilde{\mathbf{R}} = \tilde{x}/p$. This is the (gross) return on the risky asset, meaning 1 + rate of return.
- Set $R_f = 1 + r_f$. This is the (gross) risk-free return.
- The FOC is

$$\mathrm{E}[u'(\tilde{w}^*)\{\widetilde{\mathbf{R}} - R_f\}] = 0$$

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- The FOC is

$$\mathsf{E}[u'(\tilde{w}^*)\{\widetilde{\mathbf{R}} - R_f\}] = 0$$

- In words, marginal utility at the optimum is orthogonal to the excess return.

Preview of Chapter 3

- Set $\tilde{m} = u'(\tilde{w}^*)$. The FOC is $\mathsf{E}[\tilde{m}(\widetilde{\mathbf{R}} - R_f)] = 0$.
- By the definition of covariance,

$$\mathsf{E}[\tilde{m}(\widetilde{\mathbf{R}} - R_f)] = \mathsf{E}[\tilde{m}]\mathsf{E}[\widetilde{\mathbf{R}} - R_f] + \mathsf{cov}(\tilde{m}, \widetilde{\mathbf{R}} - R_f)$$

- So, the risk premium is

$$\mathsf{E}[\widetilde{\mathbf{R}} - R_f] = -\frac{1}{\mathsf{E}[\tilde{m}]}\,\mathsf{cov}(\tilde{m}, \widetilde{\mathbf{R}})$$

- What sign should the covariance with marginal utility have?

Notation

- Single consumption good at each of two dates 0 and 1
- Date–0 wealth w_0 (in units of consumption good)
- Assets
  - Assets $i = 1, \ldots, n$
  - Date–0 prices p_i (in units of consumption good)
  - Date–1 payoffs $\tilde{x}_i$ (in units of consumption good)
- Returns
  - Returns $\widetilde{R}_i = \tilde{x}_i / p_i$ (assuming $p_i > 0$)
  - Rates of return $(\tilde{x} - p_i)/p_i = \widetilde{R}_i - 1$
  - If there is a risk-free asset ($\tilde{x}$ constant) then return is R_f
- Portfolios
  - $\theta_i =$ number of shares held in portfolio
  - $\phi_i = \theta_i p_i =$ units of consumption good invested
  - $\pi_i = \theta_i p_i / w_0 =$ fraction of wealth invested

Portfolio Choice Problem

- Choose $\theta_1, \ldots, \theta_n$ to

$$\max \ \mathrm{E}\left[u\left(\sum_{i=1}^n \theta_i \tilde{x}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^n p_i\theta_i = w_0\,.$$

- Choose $\phi_1, \ldots, \phi_n$ to

$$\max \ \mathrm{E}\left[u\left(\sum_{i=1}^n \phi_i \widetilde{R}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^n \phi_i = w_0\,.$$

- Choose $\pi_1, \ldots, \pi_n$ to

$$\max \ \mathrm{E}\left[u\left(w_0\sum_{i=1}^n \pi_i \widetilde{R}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^n \pi_i = 1\,.$$

Comments

- Short sales are allowed ($\theta_i < 0$)
- There are no margin requirements.
  - In the U.S. stock market, an investor with \$100 cash can only buy \$200 of stock (borrowing \$100).
  - In our formulation, there are no limits on borrowing, except that $\sum \theta_i \tilde{x}_i$ must be in the domain of $u(\cdot)$—for example, positive if $u = \log$.
  - In real markets, collateral (margin) also has to be posted against short sales, but we do not require that in our formulation.
- Here, we take date–0 consumption and investment as given and optimize over the portfolio. We can also optimize over date–0 consumption and investment.
- We can sometimes allow for other non-portfolio income $\tilde{y}$ at date–1 (for example, labor income).

First-Order Condition

- Lagrangean:

$$\mathsf{E}\left[u\left(\sum_{i=1}^n \theta_i \tilde{x}_i\right)\right] - \lambda\left(\sum_{i=1}^n p_i\theta_i - w_0\right)$$

- Assume interior optimum and assume can interchange differentiation and expectation to obtain

$$(\forall\, i) \quad \mathsf{E}\left[u'\left(\sum_{i=1}^n \theta_i \tilde{x}_i\right)\tilde{x}_i\right] = \lambda p_i$$

- If $p_i > 0$,

$$\mathsf{E}\left[u'\left(\sum_{i=1}^n \theta_i \tilde{x}_i\right)\widetilde{R}_i\right] = \lambda$$

First-Order Condition cont.

- If $p_i > 0$ and $p_j > 0$,

$$\mathsf{E}\left[u'\left(\sum_{i=1}^{n}\theta_i\tilde{x}_i\right)(\widetilde{R}_i - \widetilde{R}_j)\right] = 0$$

- In words: marginal utility at the optimal wealth is orthogonal to excess returns.
  - A return is the payoff of a unit-cost portfolio.
  - An excess return is the payoff of a zero-cost portfolio (for example, a difference of returns).
- Why? Investing a little less in asset j and a little more in asset i (or the reverse) cannot increase expected utility at the optimum.

Some Results for One Risky Asset

Go Long if Risk Premium is Positive

- Let $\phi =$ amount invested in risky asset, so $w_0 - \phi$ is invested in risk-free asset. Let $\mu = \mathsf{E}[\widetilde{R}]$ and $\sigma^2 = \mathsf{var}(\widetilde{R})$.
- Date–1 wealth is

$$\tilde{w} = (w_0 - \phi)R_f + \phi\widetilde{R} = w_0 R_f + \phi(\widetilde{R} - R_f)$$

- We will show: $\mu > R_f \Rightarrow \phi^* > 0$ (by symmetry, $\mu < R_f \Rightarrow \phi^* < 0$).

Proof

We want to compare $\mathsf{E}[u(w_0 R_f + \phi(\widetilde{R} - R_f)]$ to $u(w_0 R_f)$.

Define $\overline{w} = w_0 R_f + \phi(\mu - R_f)$ and $\tilde{\varepsilon} = \phi(\widetilde{R} - \mu)$, so

$$w_0 R_f + \phi(\widetilde{R} - R_f) = \overline{w} + \tilde{\varepsilon}\,.$$

Define π by

$$u(\overline{w} - \pi) = \mathsf{E}[u(\overline{w} + \tilde{\varepsilon})]\,.$$

The variance of $\tilde{\varepsilon}$ is $\phi^2\sigma^2$, so by second-order risk aversion,

$$\pi \approx \frac{1}{2}\alpha(\overline{w})\phi^2\sigma^2 < (\mu - R_f)\phi$$

when $\phi > 0$ and small, so

$$u(\overline{w} - \pi) > u(\overline{w} - (\mu - R_f)\phi) = u(w_0 R_f)$$

DARA Implies Risky Asset is a Normal Good

- Normal good: demand rises when income (wealth) rises. Inferior: demand falls when income (wealth) rises.
- A single risky asset with $\mu > R_f$ is a normal good if the investor has decreasing absolute risk aversion.
- Proof: The FOC is

$$\mathsf{E}[u'(\tilde{w})(\widetilde{R} - R_f)] = 0$$

Differentiate it:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}w_0}\mathsf{E}[u'(w_0 R_f + \phi(\widetilde{R} - R_f))(\widetilde{R} - R_f)] \\ &= \mathsf{E}[u''(\tilde{w})\{R_f + \phi'(w_0)(\widetilde{R} - R_f)\}(\widetilde{R} - R_f)] \end{aligned}$$

Rearrange as

$$\phi'(w_0) = -\frac{R_f \mathsf{E}[u''(\tilde{w})(\widetilde{R} - R_f)]}{\mathsf{E}[u''(\tilde{w})(\widetilde{R} - R_f)^2]}\,.$$

Can show: DARA $\Rightarrow$ $\phi' > 0$.

CARA-Normal with Single Risky Asset

Assume CARA utility $\mathsf{E}[-\mathrm{e}^{-\alpha\tilde{w}}]$. Assume $\widetilde{R} \sim$ normal (μ, σ). Then $\tilde{w}$ is normally distributed.

Recall: If $\tilde{x}$ is normally distributed with mean μ_x and std dev σ_x, then

$$\mathsf{E}[\mathrm{e}^{\tilde{x}}] = \mathrm{e}^{\mu_x + \sigma_x^2/2}$$

Given an investment ϕ in the risky asset, $-\alpha\tilde{w}$ is normal with mean $-\alpha w_0 R_f - \alpha\phi(\mu - R_f)$ and std dev $\alpha\phi\sigma$. Hence,

$$\mathsf{E}[-\mathrm{e}^{-\alpha\tilde{w}}] = -\mathrm{e}^{-\alpha[w_0 R_f + \phi(\mu - R_f) - \alpha\phi^2\sigma^2/2]}$$

Thus,

$$w_0 R_f + \phi(\mu - R_f) - \alpha\phi^2\sigma^2/2$$

is the certainty equivalent (mean minus one-half risk aversion times variance).

CARA-Normal cont.

Optimal portfolio maximizes the certainty equivalent. Therefore, the optimum is

$$\phi^* = \frac{\mu - R_f}{\alpha\sigma^2}$$

The optimal fraction of wealth to invest is

$$\pi^* = \frac{\mu - R_f}{(\alpha w_0)\sigma^2}$$

Usually assume αw_0 is between 1 and 10.

Multiple Risky Assets

Portfolio Mean and Variance

- $\widetilde{\mathbf{R}} =$ n–vector of risky asset returns
- $\mu =$ n–vector of expected returns
- $\phi =$ n–vector of investments in consumption good units
- $\pi = (1/w_0)\phi$
- $\iota =$ n–vector of 1's
- $\Sigma =$ $n \times n$ covariance matrix, $\Sigma_{ij} = \text{cov}(\widetilde{R}_i, \widetilde{R}_j)$

$$\Sigma = \mathsf{E}[(\widetilde{R} - \mu)(\widetilde{R} - \mu)']$$

- date–1 wealth $\tilde{w} = w_0 R_f + \phi'(\widetilde{\mathbf{R}} - R_f \iota)$
- expected wealth $\overline{w} = w_0 R_f + \phi'(\mu - R_f \iota)$
- variance of wealth $= \phi' \Sigma \phi$. Proof:

$$\mathsf{E}[(\tilde{w} - \overline{w})^2] = \mathsf{E}[\{\phi'(\widetilde{\mathbf{R}} - \mu)\}^2] = \mathsf{E}[\phi'(\widetilde{R} - \mu)(\widetilde{R} - \mu)'\phi] = \phi'\Sigma\phi$$

Diversification

- Portfolio variance is

$$\pi' \Sigma \pi = \sum_{i=1}^{n} \pi_i^2 \operatorname{var}(\widetilde{R}_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \pi_i \pi_j \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_j)$$

- We can generally make $\sum_{i=1}^{n} \pi_i^2 \operatorname{var}(\widetilde{R}_i)$ small by diversifying, if there are many assets.
- Suppose for example that the risky assets are uncorrelated and have the same variance σ^2 ($\Sigma = \sigma^2 I$). Then

$$\pi' \Sigma \pi = \sigma^2 \sum_{i=1}^{n} \pi_i^2$$

  Among portfolios fully invested in risky assets (π_i sum to 1), this variance is minimized at $\pi_i = 1/n$ and

$$\pi' \Sigma \pi = \sigma^2 \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 = \frac{\sigma^2}{n} \to 0 \quad \text{as } n \to \infty$$

CARA-Normal with Multiple Risky Assets

- Certainty equivalent is mean minus one-half risk aversion times variance:

$$w_0 R_f + \phi'(\mu - R_f \iota) - \frac{1}{2}\alpha\phi'\Sigma\phi$$

- FOC is

$$\mu - R_f \iota - \alpha\Sigma\phi = 0 \,.$$

- Optimum is

$$\phi^* = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f \iota)$$

  Note no wealth effects.

- Similar form to single risky asset case. Optimum investment in each asset depends on its covariances with other assets unless Σ is diagonal.

Wealth Expansion Paths

Portfolio Return

- With a risk-free asset, date-1 wealth is

$$\sum_i \phi \widetilde{\mathbf{R}}_i + \left(w_0 - \sum_i \phi_i \right) R_f$$

- Divide by w_0 and write $\pi_i = \phi_i / w_0$.
- Date-1 wealth is

$$w_0 \left[\sum_i \pi \widetilde{\mathbf{R}}_i + \left(1 - \sum_i \pi_i \right) R_f \right] := w_0 \widetilde{\mathbf{R}}_p$$

- In words, initial wealth times the (gross) portfolio return.

How does a Log Utility Investor's Portfolio Depend on Wealth?

- Utility is

$$\log(w_0\widetilde{\mathbf{R}}_p) = \log w_0 + \log \widetilde{\mathbf{R}}_p = \log w_0 + \log\left(\sum_i \pi\widetilde{\mathbf{R}}_i + \left(1 - \sum_i \pi_i\right) R_f\right)$$

- The optimal portfolio maximizes expected log of portfolio return and is independent of initial wealth.
- Optimal dollar investments are proportional to initial wealth: $\phi_i^* = w_0\pi_i^*$.
- Optimal shares are also proportional to initial wealth: $\theta_i^* = \phi_i^*/p_i = w_0\pi_i^*/p_i$.

Power Utility

- Utility is
$$\frac{1}{1-\rho}(w_0\widetilde{\mathbf{R}}_p)^{1-\rho} = w_0^{1-\rho} \times \frac{1}{1-\rho}\widetilde{\mathbf{R}}_p^{1-\rho}$$
- So $w_0^{1-\rho}$ is a positive constant that multiplies the expected utility of the portfolio return.
- Optimal portfolio π^* maximizes expected utility of portfolio return and is independent of initial wealth.
- Optimal dollar investments are proportional to initial wealth: $\phi_i^* = w_0\pi_i^*$.
- Optimal shares are also proportional to initial wealth: $\theta_i^* = \phi_i^*/p_i = w_0\pi_i^*/p_i$.

CARA Utility

- Stick with \$ investments. Date-1 wealth is

$$\sum_i \phi\widetilde{\mathbf{R}}_i + \left(w_0 - \sum_i \phi_i\right) R_f = w_0 R_f + \phi'(\widetilde{\mathbf{R}} - R_f\iota)$$

- Here, ϕ is vector of ϕ_i, $\widetilde{\mathbf{R}}$ is vector of $\widetilde{\mathbf{R}}_i$ and ι is vector of 1's.
- Utility is

$$-\mathrm{e}^{-\alpha\tilde{w}_1} = \mathrm{e}^{-\alpha w_0 R_f} \times \left(-\mathrm{e}^{-\alpha\phi'(\widetilde{\mathbf{R}} - R_f\iota)}\right)$$

- Optimal dollar investments are independent of initial wealth (absence of wealth effects).
- Optimal shares are also independent of initial wealth.

LRT Utility

- Optimal dollar investments are affine in initial wealth: $\phi_i^* = a_i + b_i w_0$.
- Optimal shares are also affine in initial wealth (divide a_i and b_i by p_i).
- We say "wealth expansion paths are linear."
- Slope coefficient depends on cautiousness parameter. (Recall LRT means $\tau = A + Bw$ and B is called the cautiousness parameter.)
- CARA investors have horizontal (zero slope) expansion paths.
- CRRA investors with same relative risk aversion have parallel expansion paths (same b_i).

Euler Equation

Time-Additive Utility and the Euler Equation

- Date–0 and date–1 consumption. Utility function $v(c_0, c_1)$. Assume time-additive utility

$$v(c_0, c_1) = u(c_0) + \delta u(c_1)$$

- Consumption/investment problem: choose $c_0, \phi_1, \ldots, \phi_n$ to

$$\max \; u(c_0) + \mathrm{E}\left[\delta u\left(\sum_{i=1}^n \phi_i \widetilde{R}_i\right)\right] \quad \text{subject to} \quad c_0 + \sum_{i=1}^n \phi_i = w\,.$$

- FOC: $u'(c_0) = \lambda$ and

$$(\forall\, i) \quad \mathrm{E}\left[\delta u'\left(\sum_{i=1}^n \phi_i \widetilde{R}_i\right)\widetilde{R}_i\right] = \lambda$$

- So

$$(\forall\, i) \quad \mathrm{E}\left[\frac{\delta u'\left(\sum_{i=1}^n \phi_i \widetilde{R}_i\right)}{u'(c_0)}\widetilde{R}_i\right] = 1$$

Exercise 2.6

With time-additive CRRA utility, the elasticity of intertemporal substitution is the reciprocal of relative risk aversion.

Definition of EIS: for a utility function $v(c_0, c_1)$.

$$\text{MRS} = \frac{\partial v / \partial c_0}{\partial v / \partial c_1}$$

$$\text{EIS} = \frac{\mathrm{d} \log(c_1/c_0)}{\mathrm{d} \log MRS}$$

Set $x = c_1/c_0$. Assume time-additive CRRA utility:

$$v(c_0, c_1) = \frac{1}{1-\rho} c_0^{1-\rho} + \frac{\delta}{1-\rho} c_1^{1-\rho}$$

Then MRS $= -\log \delta + \rho \log x$. So, $\mathrm{d} \log \text{MRS} / \mathrm{d} \log x = \rho$. This implies

$$\text{EIS} = \frac{1}{\rho}$$